# Topology

By,

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II YEAR – III SEMESTER
COURSE CODE: 7MMA3C2

## CORE COURSE-X–TOPOLOGY – I

**Unit I**

Topological Spaces – Basis of a topology – the order topology – the product topology on XxY– the subspace topology – closed sets and limit points.

**Unit II**

Continuous functions – the product topology – the metric topology – the quotient topology.

**Unit III**

Connected spaces – connected sets in the real line – components and path components – local connectedness.

**Unit IV**

Compact spaces – compact sets in the real line – limit point compactness.

**Unit V**

The countability axioms – the separation axioms – the Urysohn's lemma – the Uryshon's metrization theorem.

**Text Book**

James R.Munkres, Topology a first course, Prentice Hall of India Pvt. Ltd.,New Delhi (1987).

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|---|---|---|
| Chapter II | : | (Sections 2.1 to 2.10) |
| Chapter III | : | (Sections 3.1 to 3.4) |
| Chapter IV | : | (Sections 3.5 to 3.7) |
| Chapter V | : | (Sections 4.1 to 4.4) |

**Books for Supplementary Reading and Reference:**

1. James Dugundji, Topology, Prentice Hall of India, New Delhi, 1975.
2. George F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill Book Co., 1963.

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# Topology

## UNIT 1

Core course - Topology - I
Course code : 7MMA3C2

Unit: I

Topological spaces - Basis of a topology - the order topology - the product topology on X×Y - the subspace topology - closed sets and limit points

# MSC MATHEMATICS

## II YEAR

## TOPOLOGY - I

(M. Sirin Hasina).

Core course - Topology - I

course code : 7MMA3C2

Unit: I

Topological spaces - Basis of a topology - the order topology - the product topology on $X \times Y$ - the subspace topology - closed sets and limit points

Unit - II

continuous functions - the product topology - the metric topology - the quotient topology

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connected spaces - connected sets in the real line - components and path components - local connectedness.

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chapter II : sections 2.1 to 2.10

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Unit - 1

## Topological space.

Defn:-

Let X be a non-empty set and $\mathcal{T}$ be the collection of subsets of X. then $\mathcal{T}$ is said to be topology on X. if it satisfies the following axioms,

i, X and $\phi$ are in $\mathcal{T}$.

ii, The arbitrary union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

ie, if $A_\alpha \in \mathcal{T}$, $\alpha \in J$

Then $\bigcup_\alpha A_\alpha \subset \mathcal{T}$.

iii, The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

ie, if $A_i \in \mathcal{T}$, $i = 1, 2 \dots n$

Then $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$.

The set X together with the topology $\mathcal{T}$ is called a topological space X. It is denoted by $(X, \mathcal{T})$.

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Defn:-

Let $(X, \mathcal{T})$ be the topological space then the elements of $\mathcal{T}$ are called open sets.

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Note :-

X and $\phi$ are both open.

Arbitrary union of open sets is also open.

finite intersection of open sets is open.

We may define more than one topology on any non-empty set X.

Ex:.

1, Let $X$ be a non-empty set $\mathcal{T} = \{X, \phi\}$ is a topology on $X$. Then this topology is called the indiscrete topology.

2, Let $X$ be a non-empty set and $T$ be the collection of all subsets of $X$. then $T$ is a topology on $X$. This topology is called discrete topology.

3, Let $X = \{a, b, c\}$ We may define the following topology on $X$.

$X = \{a, b, c\}$

$\mathcal{T}_1 = \{X, \phi\}$

$\mathcal{T}_2 = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$

$\mathcal{T}_3 = \{X, \phi, \{a\}\}$

$\mathcal{T}_4 = \{X, \phi, \{c, a\}\}$

$\mathcal{T}_5 = \{X, \phi, \{b, c\}, \{b\}, \{c\}\}$

$\mathcal{T}_6 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$

$\mathcal{T}_7 = \{X, \phi, \{c\}, \{a\}, \{a, c\}, \{a, b\}\}$.

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finite complement topology:.

(t) Let $X$ be a non-empty set and Let $\mathcal{T}_f = \{U / X - U$ is finite or all of $X\}$. Then prove that $\mathcal{T}_f$ is a topology on $X$.

Proof:.

i, $X - X = \phi$ is finite.

$\therefore X \in \mathcal{T}_f$

$X - \phi = X$

$\phi \in \mathcal{T}_f$

ii, Let $U_\alpha \in \mathcal{T}_f \quad \alpha \in J$

To prove that, $\bigcup_\alpha U_\alpha \in \mathcal{T}_f$.

Since $U_\alpha \in \mathcal{T}_\alpha$.

$\Rightarrow X - U_\alpha$ is Countable or all of $X \longrightarrow (1)$

$X - \bigcup_\alpha U_\alpha = \left(\bigcup_\alpha U_\alpha\right)^c = \cap U_\alpha{}^c$

$= \cap (X - U_\alpha) \qquad$ [by (1)]

$=$ Countable or all of $X$.

Since $U_\alpha \in \mathcal{T}_f$.

$X - U_\alpha$ is finite or all of $X \rightarrow (1)$

$X - \bigcup_\alpha U_\alpha = (\bigcup_\alpha U_\alpha)^c = \cap U_\alpha{}^c$

$= \cap (X - U_\alpha)$ [by (1)]

$=$ finite or all of $X$. ~~→~~ (1)

$\bigcup_\alpha U_\alpha \in \mathcal{T}_f$

iii) Let $U_1, U_2 \ldots U_n$ be the finite number of set in $\mathcal{T}_f$.

To prove, $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_f$.

Since $U_i \in \mathcal{T}_f \Rightarrow X - U_i$ is finite or all of $X. \rightarrow (2)$

Consider,

$$X - \bigcap_{i=1}^{n} U_i = (\bigcap_{i=1}^{n} U_i)^c$$
$$= \bigcup_{i=1}^{n} U_i{}^c$$
$$= \bigcup_{i=1}^{n} (X - U_i).$$

$=$ finite or all of $X$ by (2)

$\bigcap_{i=1}^{n} U_i \in \mathcal{T}_f$.

$\mathcal{T}_f$ is topology on $X$.

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Countable Complement topology :-

1). Let $X$ be a non-empty set and. Let $\mathcal{T}_c = \{U / U \subset X, X - U$ is countable or all of $X\}$. Then prove that $\mathcal{T}_c$ is a topology on $X$.

Pf:-

i)

$X - X = \phi$ is countable.

$X \in \mathcal{T}_c$.

$X - \phi = X$

$\phi \in \mathcal{T}_c$.

ii)

Let $U_\alpha \in \mathcal{T}_c$, $\alpha \in J$.

To prove that, $\bigcup_\alpha U_\alpha \in \mathcal{T}_c$.

Since $U_\alpha \in \mathcal{T}_c$.

$\Rightarrow X - U_\alpha$ is countable or all of $X. \rightarrow (1)$

$X - \bigcup U_\alpha = (\bigcup_\alpha U_\alpha)^c = \cap U_\alpha{}^c$

$= \cap (X - U_\alpha)$ [by (1)]

$=$ countable or all of $X$.

$\therefore \bigcup_\alpha U_\alpha \in \mathcal{T}_c$.

iii) Let $U_1, U_2 \ldots U_n$ be the finite number of set in $\mathcal{T}_c$.

To prove, $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_c$.

Since $U_i \in \mathcal{T}_c \Rightarrow X - U_i$ is countable or all of $X$ $\mapsto (2)$

Consider,
$$X - \bigcap_{i=1}^{n} U_i = \left(\bigcap_{i=1}^{n} U_i\right)$$
$$= \bigcup_{i=1}^{n} U_i^c$$
$$= \bigcup_{i=1}^{n} (X - U_i).$$

$=$ countable or all of $X$. by (2).

$\therefore \bigcap_{i=1}^{n} U_i \in \mathcal{T}_c$.

$\therefore \mathcal{T}_c$ is topology on $X$.

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**Defn:**

## Comparison of topology:.

Let $\mathcal{T}$ and $\mathcal{T}'$ be any two topologies on $X$. If $\mathcal{T}' \supset \mathcal{T}$, then we say that topology $\mathcal{T}'$ is finer or stronger or larger than $\mathcal{T}$.

If $\mathcal{T}'$ properly contains $\mathcal{T}$. the we say that $\mathcal{T}'$ strictly finer than $\mathcal{T}$.

We also say that $\mathcal{T}$ is coarser or weaker or smaller than $\mathcal{T}'$. If $\mathcal{T}$ is properly contained in $\mathcal{T}'$. Then we say that $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$.

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**Defn:**

## Basis for topology.

If $X$ is a set, a basis for topology on $X$. is a collection $\mathcal{B}$ of subsets of $X$ such that,

i, foreach $x \in X$, there exists atleast one basis element $B$ containing $x$.

ii) If $x$ belongs to the intersection of $B_1$ and $B_2$ then there exists a basis element $B_3$ containing $x$ such that $B_3 \subset B_1 \cap B_2$.

## Topology generated by a basis. (5)

Open Set:-

A Subset $U$ of $X$ is said to be open in $X$, if for each $x \in U$. there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

If a topology $\tau$ is defined by the above method then we say the topology $\tau$ is generated by the basis $\mathcal{B}$.

Note:-

Each element of $\mathcal{B}$ is open in $X$. [Under above defn]. So that $\mathcal{B}$ is a subset of $\tau$.

Ex: 1

Let $\mathcal{B}$ be denote the collection of interior point of the circular region in $\mathbb{R}^2$. then prove that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$.

pf:-

i) Suppose $(x,y) \in \mathbb{R}^2$.

We can have one $B \in \mathcal{B}$. such that $(x,y) \in B$.

$B_1, B_2 \in \mathcal{B}$ and $(x,y) \in B_1 \cap B_2$.

Let $(x_1, y_1)$ be a centre and $r_1$ be the radius of $B_1$.

and $(x_2, y_2)$ be a Centre and $r_2$ be the radius of $B_2$.

$$r = \min\{ r_1 - d((x,y), (x_1,y_1)),\ r_2 - d((x,y),(x_2,y_2)) \}$$

$B_3$ = The interior of the circular region with Centre $(x,y)$ and radius $r$.

$\therefore$ We get $(x,y) \in B_3$ subset of $B_1 \cap B_2$.

$\therefore \mathcal{B}$ is a basis for a topology in $\mathbb{R}^2$

Ex: 2

Let $\mathcal{B}$ denote the collection of all rectangular region in the plane or in $\mathbb{R}^2$. Where the rectangular pairsides $\parallel^{d}$ to the Co-ordinate axis. The $\mathcal{B}$ is a basis for a topology in $\mathbb{R}^2$.

Let $\mathcal{B}$ be the collection of all singleton subsets of set X. then $\mathcal{B}$ is a basis for the discrete topology on x.

pf:

i, Let $B = \{x\} \in \mathcal{B}$.

for each $x$ in x, there exists $B = \{x\} \in \mathcal{B}$ such that, $x \in B$.

ii,

$B_1 = \{y\}$, $B_2 = \{y\}$.

$\therefore y \in B_1 \cap B_2$.

then $B_1 \cap B_2 = B_3$.

$\therefore$ There exist $B_1 \in \mathcal{B}$ such that $y \in B_3 \subset B_1 \cap B_2$.

Remark:-

Basis is always a subset of $\mathcal{T}$.

Remark:.

PT topology generated by a basis $\mathcal{T}(\mathcal{B})$ satisfies the axioms of the topology.

pf:

Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on x.

and $\mathcal{T}(\mathcal{B}) = U \subset x /$ for each $x \in U$ there exists $B \subset \mathcal{B}$ such that $x \in$ ___

claim, ~~$\mathcal{T}$~~ $\mathcal{T}(\mathcal{B})$ is topology.

i, Obviously, $\phi \in \mathcal{T}(\mathcal{B})$

Again for each $x \in X$, there exists $B \subset \mathcal{B}$ such that,

$x \in B \subset x \Rightarrow x \in \mathcal{T}(\mathcal{B})$

ii, Let $\{U_\alpha / \alpha \in J\}$ be a subfamily of $\mathcal{T}(\mathcal{B})$

To prove, $\bigcup_\alpha U_\alpha \in \mathcal{T}(\mathcal{B})$

let $x \in \bigcup_\alpha U_\alpha \Rightarrow x \in U_\alpha$ for some $\alpha$.

$\Rightarrow$ there exist $B_\alpha \in \mathcal{B}$ such that $x \in B_\alpha \subset U_\alpha$.

$\Rightarrow x \in B_\alpha \subset \bigcup_\alpha U_\alpha$

$\Rightarrow \bigcup_\alpha U_\alpha \in \mathcal{T}(\mathcal{B})$.

iii) To prove, $\bigcap_{i=1}^{n} U_i \in \mathcal{T}(\mathcal{B})$

Prove this result by induction on $n$.

Let $U_1, U_2 \in \mathcal{T}(\mathcal{B})$

claim, $U_1 \cap U_2 \in \mathcal{T}(\mathcal{B})$

Let $x \in U_1 \cap U_2 \Rightarrow x \in U_1$ and $x \in U_2$.

$x \in U_1 \Rightarrow$ there exist some $B_1 \in \mathcal{B}$ such that

$x \in B_1 \subset U_1$

$x \in U_2 \Rightarrow$ there exist some $B_2 \in \mathcal{B}$ such that

$x \in B_2 \subset U_2$.

$\Rightarrow x \in B_1 \cap B_2 \subset U_1 \cap U_2$

$\therefore U_1 \cap U_2 \in \mathcal{T}(\mathcal{B})$

ie, the result is true for $n=2$.

Assume that the result is true for $n-1$.

ie, $\bigcap_{i=1}^{n-1} U_i \in \mathcal{T}(\mathcal{B})$

Let $U = \bigcap_{i=1}^{n-1} U_i$ , $U_n \subset \mathcal{T}(\mathcal{B})$.

By induction hypothesis, $U \cap U_n \in \mathcal{T}(\mathcal{B})$

ie, $\bigcap_{i=1}^{n-1} U_i \cap U_n \in \mathcal{T}(\mathcal{B})$

ie, $\bigcap_{i=1}^{n} U_i \in \mathcal{T}(\mathcal{B})$

$\therefore \mathcal{T}(\mathcal{B})$ is topology on $X$.

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Lemma 1.2.1.

(*) Let $X$ be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$, then $\mathcal{T}$ equals the collection of all union of elements of $\mathcal{B}$.

[or] Any member of $\mathcal{T}$ can be expressed as the union of $\mathcal{B}$.

Pf:-

Let $\mathcal{B}$ be a basis for the topology on $X$ and $\mathcal{T}$ is the topology generated by the basis $\mathcal{B}$.

ie, $\mathcal{T} = \{U \subset X \,/\, \forall x \in U$ there exist $B \in \mathcal{B}$ such that $x \in B \subset U\}$

To prove,

$$\mathcal{T} = \{\bigcup_{\alpha} B_\alpha / B_\alpha \in \mathcal{B}\}$$

Let $U$ be an element in $\mathcal{T}$.

$\Rightarrow U$ is open in $X$.

For each $x \in U$, there exist $B_x \in \mathcal{B}$ such that, $x \in B_x \subset U$.

$B_x \subset U \; \forall x \in U$, $\Rightarrow \bigcup B_x \subset U \rightarrow (1)$

By defn. of $\mathcal{T}$,

$\forall x \in U$, there exist $B_x \in \mathcal{B}$ such that $x \in B_x$.

$\Rightarrow x \in \bigcup B_x$

$\therefore U \subset \bigcup B_x \rightarrow (2)$

From (1) and (2),

$U = \bigcup B_x \in \{\bigcup_\alpha B_\alpha\}$

$\therefore \mathcal{T} \subset \{\bigcup B_\alpha\} \rightarrow (3)$

Claim, $\{\bigcup B_\alpha\} \subset \mathcal{T}$.

Let $B_\alpha \in \mathcal{B} \; \forall \alpha$.

$\Rightarrow B_\alpha \in \mathcal{T}$

$\bigcup B_\alpha \in \mathcal{T}$.

[$\because \mathcal{T}$ is topology]

$\Rightarrow \{\bigcup B_\alpha\} \subset \mathcal{T} \rightarrow (4)$

From (3) & (4)

$\mathcal{T} = \{\bigcup_\alpha B_\alpha\} / B_\alpha \in \mathcal{B}\}$.

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Lemma 1.2.2.

Let $\mathcal{B}$ and $\mathcal{B}'$ be any two basis for the topology $\mathcal{T}$ and $\mathcal{T}'$ on a set $X$. The following are equivalent.

i, $\mathcal{T}'$ is finer than $\mathcal{T}$

ii, for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing $x$, there exists a basis element $B' \in \mathcal{B}'$ such that $B' \subset B$.

Pf:

(i) $\Rightarrow$ (ii)

Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$.

ie, $\mathcal{T}' \supset \mathcal{T}$.

To prove,

for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing $x$, there exist a basis element $B' \in \mathcal{B}'$ such that $B' \subset B$.

Given $x \in X$ Let $B \in \mathcal{B}$ and $x \in B$, but $B \subset \mathcal{T}$.

$\therefore B \in \mathcal{T} \subset \mathcal{T}'$

$\Rightarrow B \in \mathcal{T}'$ (By gn. condition)

ie, $B$ is $\mathcal{T}'$-open.

$\therefore \mathcal{B}$ is a basis for $\mathcal{T}'$.

But $\mathcal{B}'$ is a basis for $\mathcal{T}'$.

For each $x \in B$ there exists $B' \in \mathcal{B}' \ni: x \in B' \subset B$.

$\therefore B' \subset B$.

claim, (ii) $\Rightarrow$ (i)

To prove,

$\mathcal{T}' \supset \mathcal{T}$

ie, $\mathcal{T} \subset \mathcal{T}'$

Let $U \in \mathcal{T}$

$\therefore U$ is $\mathcal{T}$-open.

Then for each $x \in U$ there exist $B \in \mathcal{B} \ni: x \in B \subset U$.

By given condition (ii) there exist $B' \in \mathcal{B}' \ni:$ $x \in B' \subset B \subset U$

Thus for each $x \in U$ there exist $B' \in \mathcal{B}' \ni: B' \subset U$.

$\Rightarrow U$ is $\mathcal{T}'$-open

$\Rightarrow U \in \mathcal{T}'$

$\Rightarrow \mathcal{T} \subset \mathcal{T}'$

ie, $\mathcal{T}'$ is finer than $\mathcal{T}$.

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**Lemma 1.2.3.**

Let $X$ be a topological space and Let $\mathcal{C}$ is a collection of open sets of $X$ such that for each $x$ in $X$ ~~and for each~~ open sets $U$ of $X$ there is an element $C$ of $\mathcal{C}$ such that $x \in C \subset U$. then $\mathcal{C}$ is a basis for the topology on $X$.

Pf:-

Let $\mathcal{C}$ be a collection of open subsets of $X$ for every $U$ of $X$ and for every $x \in X$.

there exists $C \in \mathcal{C}$ such that $x \in C \subset U$

(10)

Step 1 :-

claim, $\mathcal{C}$ is a basis

We know that,

$X$ is open in $X$

$\Rightarrow x \in X$ there exist an element $C (= X) \in \mathcal{C}$ such that $x \in C \subset X$

Let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$

$C_1, C_2 \in \mathcal{C} \Rightarrow C_1, C_2$ are open in $X$.

~~for~~ $\Rightarrow C_1 \cap C_2$ is also open in $X$

for every $x \in C_1 \cap C_2$ there exist an element $C_3 = C_1 \cap C_2 \subset \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

By condition (i) and (ii) we have $\mathcal{C}$ is a basis for a topology. $\rightarrow$ (1)

Step: 2

Let $\mathcal{T}$ be the collection of open sets of $X$.

claim,

The topology $\mathcal{T}'$ generated by $\mathcal{C}$ coincide with the topology $\mathcal{T}$ of $X$.

To prove that,

$\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T}' \subset \mathcal{T}$

Let $U \in \mathcal{T}$ and $x \in U$ by hypothesis there exists an element.

$C \in \mathcal{C}$ such that $x \in C \subset U \Rightarrow U \in \mathcal{T}'$

[$\because \mathcal{T}'$ is the topology generated by $\mathcal{C}$]

$\mathcal{T} \subset \mathcal{T}' \rightarrow$ (2)

Again let $W \in \mathcal{T}'$

Since $\mathcal{C}$ is the basis for $\mathcal{T}'$

$W$ equals the union of elements of $\mathcal{C} \rightarrow$ (3)

By hypothesis,

$\mathcal{C}$ is a collection of open set of $X$.

$\Rightarrow \mathcal{C} \in \mathcal{T}$

$\Rightarrow$ All elements of $\mathcal{C} \in \mathcal{T}$

$\Rightarrow W \in \mathcal{T}$ [by (3)]

$\therefore \mathcal{T}' \subset \mathcal{T} \longrightarrow$ (4)

By (2) and (4) we have

$\mathcal{T} = \mathcal{T}'$

$\mathcal{C}$ is the basis for the topology of X.

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Lemma 1.2.4

The lower limit topology $\mathcal{T}'$ on $\mathbb{R}$ is strictly finer than the standard topology $\mathcal{T}$.

Pf:

Given $\mathcal{T}$ is the standard topology and $\mathcal{T}'$ is the lower limit topology on $\mathbb{R}$.

To prove,

$\mathcal{T}'$ is finer than $\mathcal{T}$.

ie, to prove $\mathcal{T}' \supset \mathcal{T}$.

Let $\mathcal{B} = \{(a,b) / a < b,\ a,b \in \mathbb{R}\}$ be a basis for $\mathcal{T}$.

$\mathcal{B}' = \{[a,b) / a,b \in \mathbb{R}'\}$ be a basis for $\mathcal{T}'$.

Let $x \in \mathbb{R}$ and $(a,b) \in \mathcal{B}$ such that $x \in (a,b)$.

We can find a half open intervals $[x,b) \in \mathcal{B}'$ such that $[x,b) \subset (a,b)$.

By lemma 1.2.2, $\mathcal{T}' \supset \mathcal{T} \longrightarrow (1)$

Also prove $\mathcal{T}$ is not finer than $\mathcal{T}'$.

Consider the $[a,b) \in \mathcal{B}'$ and also $a \in [a,b)$.

Then there exists is no open interval $B \in \mathcal{B}$ such that $a \in B \subset [a,b)$.

$\therefore [a,b)$ is not open.

$\mathcal{T}$ cannot be finer than $\mathcal{T}'$. $\longrightarrow (2)$

From (1) & (2)

$\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

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Sub basis

Defn:-

A sub basis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

ie, $\mathcal{S} = \{U \in X / U_U = X\}$

Defn:

The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection of all union of finite intersection of elements of S.

$$\mathcal{T}(\mathcal{S}) = \{ \cup \bigcap_{i=1}^{n} S_i / S_i \in \mathcal{S} \}$$

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(*) Thm 1.2.1

Let X be a set and $\mathcal{T}(\mathcal{S}) = \{ \cup \bigcap_{i=1}^{n} S_i / S_i \in \mathcal{S} \}$ Then $\mathcal{T}(\mathcal{S})$ is topology on X.

pf:

Let $\mathcal{B} = \{ \bigcap_{i=1}^{n} S_i / S_i \in \mathcal{S} \}$

Now, $\mathcal{T}(\mathcal{S})$ is the collection of union of elements of $\mathcal{B}$.

Prove that, $\mathcal{T}(\mathcal{S})$ is topology on X.

For that, it is enough to prove $\mathcal{B}$ is basis.

Let $x \in X$.

Then $x \in X = \cup S_i$ where $S_i \in \mathcal{S}$.

$\Rightarrow x \in S_i$ for some i

$\Rightarrow x \in \bigcap_{j=1}^{n} S_{ij} \in \mathcal{B}$ for some j such that $x \in B$, $B = \bigcap S_{ij}$

$\therefore$ the condition (1) for the basis is satisfied.

Let $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2$.

$B_1 = \bigcap_{i=1}^{n} S_i$, $B_2 = \bigcap_{j=1}^{m} S_j$, $S_i, S_j \in \mathcal{S}$.

$$B_1 \cap B_2 = \left[\bigcap_{i=1}^{n} S_i\right] \cap \left[\bigcap_{j=1}^{m} S_j\right]$$

$$= \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} (S_i \cap S_j) \in \mathcal{B}.$$

Let $B_3 \subset B_1 \cap B_2 \in \mathcal{B}$.

$x \in B_3 \subset B_1 \cap B_2 \in \mathcal{B}$.

$\therefore \mathcal{B}$ is a basis for $\mathcal{T}(\mathcal{S})$

$\therefore \mathcal{T}(\mathcal{S})$ is topology on X.

---

Thm 1.2.2.

P.T the collection $\mathcal{B} = \{(a,b) / a, b \in \mathbb{R}\}$ is a basis in $\mathbb{R}$.

Pf:-

Let $x \in \mathbb{R}$.

Choose $a, b \in \mathbb{R}$ such that $a < x < b$.

Now, $(a,b) \subset \mathcal{B}$.

$\therefore$ for each $x \in \mathbb{R}$. there exists $(a,b) \in \mathcal{B}$. such that $x \in (a,b)$.

i, for a basis is satisfied.

Let $B_1 = (a,b)$ $B_2 = (c,d) \in \mathbb{R}$.

$x \in B_1 \cap B_2$

Then there exist $B_3 = (c,b) \in \mathcal{B}$.

Such that $x \in (c,b) \subset (a,b) \cap (c,d)$

ie, $x \in B_3 \subset B_1 \cap B_2$

ii) for a basis satisfied.

$\mathcal{B} = \{(a,b) / a, b \in \mathbb{R}\}$ is a basis in $\mathbb{R}$.

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The ordered topology:-

A relation $<$ on a set $X$ is called a Order relation (Simple order or linear order) if it has the following properties.

i) Comparability:-

for every $x, y \in X$ for which $x \neq y$ either $x < y$ ($x$ related to $y$) or $y < x$ holds.

ii) Non-Reflexivity:-

for no $x \in X$, does the relation $x < x$ holds.

iii) Transitivity:-

If $x, y, z \in X$ such that $x < y, y < z$ then $x < z$ holds.

---

Defn:-

A simply ordered set is a set with a simply order relation. Define the topology on any simply ordered set.

Notation:-

Let $X$ be a simply ordered set with an order relation $X$. Let $a, b \in X$.

Consider the following subsets of X. (14)

$(a,b) = \{x / a < x < b\}$

$[a,b] = \{x / a \le x \le b\}$

$[a,b) = \{x / a \le x < b\}$

$(a,b] = \{x / a < x \le b\}$.

$(a,b)$ is called the open interval. $[a,b]$ is called a closed interval.

$[a,b)$, $(a,b]$ is called a half open interval.

Ex of Ordered Topology :-

The Standard topology on R is the Ordered topology on R. which usual relation. Because we know that the topology generated by the intervals $\{(a,b) / a,b \in R\}$ is a standard topology.

Consider the set R with usual order relation ($<$ or $>$). Since R has no smallest element as well as largest element.

The basis for the order topology will be only the intervals of the type $(a,b)$.

$\therefore$ Order topology for the set $(\mathbb{R}, <)$ is nothing but the standard topology on R.

Defn :- Dictionary Order relation

Consider the set $R \times R$ with a dictionary order relation with any two elements $a \times b$ and $c \times d$ in $R \times R$ is defined as $a \times b < c \times d$, if $a < c$ (or) if $a = c$, $b < d$.

Open Ray and Closed Ray :-

If $X$ is an ordered set and $a \in X$. There are 4 subsets of X that are called the rays determined by a there are,

$(a, \infty) = \{x / x > a\}$

$(-\infty, a) = \{x / x < a\}$

$[a, \infty) = \{x / x \geq a\}$

$(-\infty, a] = \{x / x \le a\}$

The set of first two are called open rays and the set of last two are called closed rays.

Defn:

Product topology on $X \times Y$.

Let $X$ and $Y$ be two topological spaces the product topology on $X \times Y$ is the topology having the basis $\mathcal{B}$ which is the collection of all sets of the form $U \times V$. Where $U$ is open subset of $X$ and $V$ is an open subset of $Y$.

ie, $\mathcal{B} = \{U \times V / U$ is open in $X$
$V$ is open in $Y\}$

First, let us verify that $\mathcal{B}$ is the basis for the topology $X$ and $Y$.

i) Pf:

Since $X$ is open $X$ and $Y$ is open in $Y$.

$X \times Y \in \mathcal{B}$

For every $x \times y \in X \times Y$.

there exists a basis $B = x \times y \in \mathcal{B}$ such that $x \times y \in B \subset X \times Y$.

ii) Let $B_1, B_2 \in \mathcal{B}$ where $B_1 = (U_1 \times V_1)$ and $B_2 = (U_2 \times V_2)$ where $U_1, U_2$ are open in $X$ and $V_1, V_2$ are open in $Y$.

$$B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2)$$
$$= (U_1 \cap U_2) \times (V_1 \cap V_2)$$

But $U_1 \cap U_2$ is open in $X$ and $V_1 \cap V_2$ is open in $Y$.

$\therefore B_1 \cap B_2 \in \mathcal{B}$.

For all ~~$x \times y \in$~~ $x \times y \subset B_1 \cap B_2$ there exists $B_3 \subset \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

ii) is satisfied.

$\therefore B$ is a basis for topology on $X \times Y$.

The topology generated by $\mathcal{B}$ is called product topology.

Ex:

We have a standard topology on $\mathbb{R}$: the standard topology, the product of this topology with itself is called the standard topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products of open sets of $\mathbb{R}$.

But the thm just proved tell us that the much smaller collection of all products $(a,b) \times (c,d)$ of open intervals in $\mathbb{R}$ will also serve as a basis for the topology of $\mathbb{R}^2$.

Thm: A.1

If $\mathcal{B}$ is the basis for the topology on $X$. $\mathcal{C}$ is basis for a topology on $Y$. Then the collection $\mathcal{D} = \{B \times C / B \in \mathcal{B}$ and $C \in \mathcal{C}\}$ is a basis for topology on $X \times Y$.

pf:

Let $B \times C \in \mathcal{D}$ be arbitrary.

$B \in \mathcal{B}$, $B$ is open in $X$.

[$\because$ Basis element is open].

$C \in \mathcal{C}$, $C$ is open in $Y$.

[$\because$ Basis element is open]

$B \times C$ belongs to the basis of the product topology and hence open in $X \times Y$.

$\therefore \mathcal{D}$ is a collection of open sets in $X \times Y$.

Let $x \times y \in X \times Y$ be arbitrary.

Let $G$ be any open set in $X \times Y$ containing $x \times y$.

$\therefore x \times y \in G$.

$\because G$ is open, $G$ contains the basis element.

$\therefore$ There exists $U$ is open in $X$ $V$ is open in $Y$ such that

$$x \times y \in U \times V \subseteq G \longrightarrow (1)$$

NOW, $U$ is open in $X$.

$\Rightarrow$ there exist $B \in \mathcal{B}$ such that $x \in B \subset U$

$V$ is open in $Y$.

$\Rightarrow$ there exist $C \in \mathcal{C}$ such that $y \in C \subset U$.

$\therefore x \times y \in B \times C \subset U \times V \subset G$ by (1)

Let $D = B \times C$ belongs to $\mathcal{D}$.

$\therefore x \times y$ belongs to $\mathcal{D} \subset G$.

Now $\mathcal{D}$ is the collection of open subsets of $X \times Y$. for every,

$x \times y \in X \times Y$ and every open set $G$ containing $x \times y$ such that,

$x \times y \in D \subset G$.

$\therefore \mathcal{D}$ is a basis.

**Defn:-**

Let $\pi: X \times Y \to X$ defined by the equation $\pi_1(x, y) = x$, Let $\pi_2 : X \times Y \to Y$ defined by the equation $\pi_2(x, y) = y$. The mapping $\pi_1, \pi_2$ are called projection of $X \times Y$ onto its first and second factors respectively. Also $\pi_1$ and $\pi_2$ are onto mappings.

**Note:-**

If $U$ is open subset of $X$. then $\pi_1^{-1}(U) = U \times Y$ which is open in $X \times Y$.

$^{\text{lly}}$ If $V$ is open subset of $Y$ then $\pi_2^{-1}(V) = X \times V$ which is open in $X \times Y$.

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (U \times Y) \cap (X \times V)$$
$$= (U \cap X) \times (Y \cap V)$$
$$\therefore \pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V.$$

**Thm 4.2.**

The collection $\mathcal{S} = \{\pi_1^{-1}(U) / U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) / V \text{ is open in } Y\}$ is a subbasis for the product topology.

**Pf:-**

Let us verify that $\mathcal{S}$ is a subbasis for a topology on $X \times Y$.

$X \times Y = \pi_1^{-1}(X)$, $X$ is open in $X$.

$= \pi_2^{-1}(Y)$, $Y$ is open in $Y$.

$\therefore X \times Y$ itself belongs to $\mathcal{S}$.

Trivially union of $\mathcal{S}$ is $X \times Y$.

$\therefore \mathcal{S}$ is subbasis for topology on $X \times Y$.

Let $\tau'$ be the topology generated by $\mathcal{S}$.

[Let $\mathcal{B}' = \{\bigcap_{i=1}^{n} s_i / s_i \in \mathcal{S}\}$]

Let $\tau$ denote the product topology on $X \times Y$.

Claim $\tau = \tau'$

Let $\mathcal{B}$ be the basis for the product topology on $X \times Y$.

$\mathcal{B} = \{U \times V / U$ is open in $X$ and $V$ is open in $Y\}$

Consider,

an arbitrary member $U \times V$ of $\mathcal{B}$ and is $\tau$ open. [ie, $B = U \times V \subset \tau$]

ie, $B = U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

= finite intersection of members of $\mathcal{S}$ of $\mathcal{B}'$. Where $\mathcal{B}'$ is the basis for $\tau'$.

$\therefore B \in \mathcal{B}' \subset \tau'$

Since $\tau'$ is the topology.

$\therefore$ Arbitrary Union of member of $\mathcal{B} \in \tau'$

$\therefore \tau \subset \tau' \to (1)$

Let $U$ be any open sets of $X$.

$\pi_1^{-1}(U) = U \times Y$ belongs to $\mathcal{B}$.

[$\because U$ is open in $X$ & $Y$ is open in $Y$]

Let $V$ be any open sets of $Y$.

$\pi_2^{-1}(V) = X \times V$ belongs to $\mathcal{B}$.

$\therefore \mathcal{S} \subset \mathcal{B} \subset \tau$.

But $\tau$ is a topology.

$\therefore$ Arbitrary union of finite intersection of members of $\mathcal{S} \in \tau$.

$\tau' \subset \tau \to (2)$

from (1) & (2) $\Rightarrow \tau = \tau'$.

Subspace topology :-

Defn :-

Let $X$ be a topological space with topology $\tau$ if $Y$ is a subset of $X$, the collection $\tau_Y = \{U \cap Y / U \in \tau\}$ or $\{Y \cap U / U \in \tau\}$ is a topology on $Y$ is called the subspace topology.

Verification for the collection $\tau_Y$ be a topology :-

First claim, $\phi, Y \in \tau_Y$

i) $\phi = \phi \cap Y$, where $\phi$ is open in $X$.

$\phi \in \tau_Y$.

||ly $Y = X \cap Y$, where $X$ is open in $X$

$\Rightarrow Y \in \tau_Y$.

ii) Let $\{V_\alpha\}$ be an arbitrary collection of members of $\tau_Y$.

Then each $V_\alpha = U_\alpha \cap Y$ where $U_\alpha \in \tau$.

Now, $\bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap Y)$

$= (\bigcup_\alpha U_\alpha) \cap Y$

$U_\alpha \in \tau \; \forall \; \alpha$ and $\tau$ be a topology on $X$.

$\Rightarrow \bigcup_\alpha U_\alpha \in \tau$

$\Rightarrow (\bigcup_\alpha U_\alpha) \cap Y \in \tau_Y$

$\Rightarrow \bigcup_\alpha V_\alpha \in \tau_Y$.

The arbitrary union of elements are in $\tau_Y$.

iii) Let $U_1 \cap Y, U_2 \cap Y \ldots U_n \cap Y$ be an element in $\tau_Y$.
Their intersection is $(U_1 \cap Y) \cap (U_2 \cap Y) \cap \ldots \cap (U_n \cap Y) = \bigcap_{i=1}^{n} U_i \cap Y$.

Now, each $U_i \in \tau$ and $\tau$ be topology in $X$.

$\Rightarrow \bigcap_{i=1}^{n} U_i \in \tau$.

$\Rightarrow (\bigcap_{i=1}^{n} U_i) \cap Y \in \tau_Y$.

$\Rightarrow \bigcap_{i=1}^{n} (U_i \cap Y) \in \tau_Y$.

thus finite intersection elements $\tau_Y$ in $\tau_Y$.

by (i) (ii) & (iii)

$\tau_Y$ is a topology on $Y$.

Note:-

The space $(Y, \tau_Y)$ is called the subspace of $Y$. The subspace topology consists of intersection of $X$ with $Y$ the subspace topology is called the relation topology for $\tau$.

Lemma
Thm: 5.1

Let $\mathcal{B}$ be a basis for the topology on $X$. Then the collection $\mathcal{B}_Y = \{B \cap Y / B \in \mathcal{B}\}$ is a basis for the subspace topology on $Y$.

Pf:-

Given $\mathcal{B}$ is a basis for the topology on $X$.

$\therefore \mathcal{B} \subset \tau \Rightarrow$ each $B \in \mathcal{B}$ is open in $X$.

[W.r.t, the subspace topology on $Y$ in $\tau_Y$.

$\tau_Y = \{U \cap Y / U \in \tau\}$

$\therefore \mathcal{B}_Y \in \tau_Y$.

$\mathcal{B}$ is a collection of open in $Y \rightarrow (1)$ ]

Consider an arbitrary open set $U \cap Y$ in $Y$. Where $U$ is open in $X$.

Let $y \in U \cap Y$

$\Rightarrow y \in U$ and $y \in Y$.

Again $\mathcal{B}$ is a basis for $\tau$ and is open in $X$ containing $y$.

$\Rightarrow$ there exist $B \in \mathcal{B}$ such that $y \in B \subset U$.

$\Rightarrow y \in B \cap Y \subset U \cap Y \longrightarrow (2)$

from (1) & (2)

$\mathcal{B}_Y = \{B \cap Y / B \in \mathcal{B}\}$ is a basis for subspace topology on $Y$.

Lemma 5.2.

Let $Y$ be a subspace of $X$. If $U$ is open in $Y$ and $Y$ is open in $X$ then $U$ is open in $X$.

Pf:

Given $Y$ is a subspace in $X$.

Also, given $U$ is open in $Y$.

$\Rightarrow U = V \cap Y$, ~~where~~ where $V$ is open in $X$.

$V$ is open in $X$, $Y$ is open in $X$.

$\Rightarrow V \cap Y$ is open in $X$

$\Rightarrow U$ is open in $X$.

$\therefore U$ is open in $Y$ and $Y$ is open in $X$.

$\Rightarrow U$ is open in $X$.

---

**Remark:**

All open subsets of $Y$ are open in $X$. if $Y$ is open in $X$.

**Ex:** For the subspace topology

Consider the topological space $\mathbb{R}$ with standard topology $\tau$.

ie, The topology generated by the basis, $\mathcal{B} = \{(a,b) / a,b \in \mathbb{R}\}$.

Let $Y = [0,1]$ be a subset in $\mathbb{R}$.

Define the basis by as for subspace topology as follow,

$$\mathcal{B}_Y = \{(a,b) \cap [0,1] / (a,b) \in \tau\}$$

ie, $\mathcal{B}_Y$ has the following types of sets.

$$(a,b) \cap [0,1] = \begin{cases} (a,b) & \text{if } a,b \in Y \\ (a,1] & \text{if } a \in Y \\ [0,b) & \text{if } b \in Y \\ \phi \text{ (or) } \mathcal{B}_Y & \text{if } a,b \in Y \end{cases}$$

The topology generated by $\mathcal{B}_Y$ is the subspace topology in $Y$.

---

**Remark:**

Since the subset $Y$ has the smallest element zero and largest element the intervals $(a,b)$, $(a,1]$, $[0,b)$ are basis elements for the ordered topology.

In the set $Y = [0,1]$ its subspace is subspace topology and its ordered topology are same.

Ex:

for subspace topology need not be same as the ordered topology.

Consider the real number system $\mathbb{R}$ with the standard topology $\tau$.

Let $Y = [0,1] \cup \{2\}$ be a subset of $\mathbb{R}$.

Then the subspace topology $\tau_Y$ is generated by the basis

$$\mathcal{B}_Y = \{(a,b) \cap Y \mid (a,b) \in \tau\}$$

We can write $\{2\} = (1,3) \cap Y$.

Because, $2 \in (1,3)$ and $2 \in [0,1] \cup \{2\}$

$\{2\} \in \mathcal{B}_Y$.

But $\{2\}$ need not be in basis $\mathcal{B}'$ for the ordered topology in $Y$.

Because, $\mathcal{B}' = \begin{cases} (a,b) \cap Y & \text{where } a,b \in Y. \\ [a,b) & \text{where } b \in Y \\ (a,b] & \text{where } a \in Y \end{cases}$

$\therefore$ subspace topology $\neq$ ordered topology.

## Closed sets and Limit point :.

Defn:

A subset $A$ of topological space $X$ is said to be closed, if its complement (ie, $X - A$) is open.

Ex:

Consider $\mathbb{R}$ with standard topology $[a,b]$ closed in $\mathbb{R}$.

$$[a,b]^c = \mathbb{R} - [a,b]$$
$$= (-\infty, a) \cup (b, \infty)$$

$(-\infty, a), (b, \infty) \in \tau$.

$\Rightarrow (-\infty, a) \cup (b, \infty) \in \tau$

$\Rightarrow [a,b]^c \in \tau$

$\Rightarrow [a,b]^c$ is open in $\mathbb{R}$.

$[a,b]$ is closed in $\mathbb{R}$. 

Thm 6.1.

Let $X$ be a topological space. then the following condition hold.

i) $X$ and $\phi$ are closed in $X$.

ii) Arbitrary intersection of closed sets is closed.

iii) finite union of closed sets is closed.

pf:-

i) Claim, $\phi$ and $X$ are closed.

Consider $\phi^c = X - \phi = X$

Which is open in $X$.

so $\phi^c$ is open in $X$.

$\Rightarrow \phi$ is closed in $X$.

Consider, $X^c = X - X = \phi$ which is open in $X$.

so $X^c$ is open in $X$

$\Rightarrow X$ is closed in $X$.

ii) Let $\{A_1, A_2 \ldots A_n, \ldots\}$

i.e. $\{A_\alpha\}_{\alpha \in J}$ be an arbitrary collection of closed set in $X$.

Claim, $\bigcap_1^\alpha A_\alpha$ is also closed in $X$.

Consider, $(\cap A_\alpha)^c = X - \cap A_\alpha$

$$= X - (A_1 \cap A_2 \cap \ldots A_\alpha \cap \ldots)$$

$$= (X - A_1) \cup (X - A_2) \cup \ldots \cup (X - A_\alpha)$$

$$= \bigcup_\alpha (X - A_\alpha) \longrightarrow (1)$$

each $A_\alpha$ is closed in $X$.

$\Rightarrow$ Each $(X - A_\alpha)$ is open in $X$.

$\Rightarrow \bigcup_\alpha (X - A_\alpha)$ is also open in $X$.

By (1) $(\cap A_\alpha)^c$ is open in $X$.

$\Rightarrow \bigcap_1^\alpha A_\alpha$ is closed in $X$.

iii) Let $A_1, A_2 \ldots A_n$ be closed in $X$.

Claim, $\bigcup_{i=1}^n A_i$ is closed in $X$.

Consider, $\left(\bigcup_{i=1}^{n} A_i\right)^c = X - \bigcup_{i=1}^{n} A_i$

$= X - (A_1 \cup A_2 \cup \dots \cup A_n)$

$= (X - A_1) \cap (X - A_2) \cap \dots \cap (X - A_n)$

$= \bigcap_{i=1}^{n} (X - A_i)$

Each $A_i$ is closed in $X$.

$\Rightarrow X - A_i$ is ~~closed~~ open in $X$.

$\Rightarrow \bigcap_{i=1}^{n} (X - A_i)$ is also open in $X$.

$\Rightarrow \left(\bigcup_{i=1}^{n} A_i\right)^c$ is open in $X$.

$\Rightarrow \bigcup_{i=1}^{n} A_i$ is closed in $X$.

---

Thm 6.2

Let $Y$ be a subspace of $X$. then a set $A$ is closed in $Y$ iff it equals the intersection of closed sets of $X$ with $Y$.

pf :-

Let $A = C \cap Y$ where $C$ is closed in $X$.

Claim, $A$ is closed in $Y$.

Given $C$ is closed in $X$.

$\Rightarrow X - C$ is open in $X$.

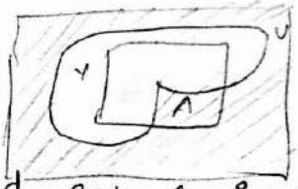
$\Rightarrow (X - C) \cap Y$ is open in $Y$ by supspace.

$\Rightarrow (X \cap Y) - (C \cap Y)$ is open in $Y$.

$\Rightarrow Y - A$ is open in $Y$

$\Rightarrow A$ is closed in $Y$.

Converseely,




Let $A$ be closed in $Y$.

Claim, $A = C \cap Y$ for some closed set $C$ in $X$.

Given $A$ is closed in $Y$.

$\Rightarrow (Y - A)$ is open in $Y$.

$\Rightarrow (\cancel{Y} - A) = U \cap Y$, where $U$ is open in $X$

$A = Y - U \cap Y$

$\Rightarrow A = X - (U \cap Y)$    $Y$ is subspace of $X$

$A = (X - U) \cap Y$.

Put $X - U = C$

$\Rightarrow X - U$ is closed in $X$.

$\Rightarrow C$ is closed in $X$.

$\therefore A = C \cap Y$. Where $C$ is closed in $X$.

Thm 6.3

Let $Y$ be a subspace of $X$ if $A$ is closed in $Y$ and $Y$ is closed in $X$. then $A$ is closed in $X$.

Pf:

Let $(X, \mathcal{J})$ be a topological space and $(Y, \mathcal{J}_Y)$ be a subspace of $(X, \mathcal{J})$.

Given $A$ is closed in $Y$.

$Y - A$ is open in $Y$.

$Y - A \in \mathcal{J}_Y \rightarrow (1)$

and given $Y$ is closed in $X$.

$X - Y$ is open in $X$.

$X - Y \in \mathcal{J} \rightarrow (2)$

$Y - A \in \mathcal{J}_Y \Rightarrow$ there exists $B \in \mathcal{J}$ such that $Y - A = B \cap Y$.

$A = Y - B \cap Y$

$A = (B \cap Y)'$

$A = Y \cap (B \cap Y)'$

$A = Y \cap (B' \cup Y')$

$A = (Y \cap B') \cup (Y \cap Y')$.

$A = (Y \cap B') \cup \phi$

$A = Y \cap B'$

By demorgan's Law $A' = (Y \cap B')' = \boxed{Y'} \cup B$. $X - Y$

$A' = (X - Y) \cup B \rightarrow (3)$

by (2), $X - Y \in \mathcal{J}$ & $B \in \mathcal{J}$.

$\therefore (X - Y) \cup B \in \mathcal{J}$

$\therefore A' \in \mathcal{J}$ [by (3)]

ie, $A'$ is open in $X$.

$X - A$ is open in $X$.

$A$ is closed in $X$.

Hence the proof.

Defn :

Interior of A ($A^\circ$)

The interior of a subset A of a topological space X is defined as the union of all open sets contained in A.

Result :-

Arbitrary union of open sets is open.

$A^\circ$ is always open and $A^\circ$ is the largest open set contained in A.

Defn :

Closure of A ($\bar{A}$)

The closure of a subset A of a topological space X is defined as the intersection of all closed sets containing A.

Result :

W.K.T arbitrary intersection of closed set is closed.

$\bar{A}$ is always a closed set and $\bar{A}$ is a smallest closed set containing A.

Prove the following,

1) If A is open then $A^\circ = A$.

2) If A is closed then $\bar{A} = A$.

Pf :

Let A be open.

$A^\circ$ = Largest open set contained in A.

$A^\circ = A$ (∵ A is open)

Let A be closed.

$\bar{A}$ = Smallest closed set containing A.

$\bar{A} = A$ [∵ A is closed]

In general case, always.

1) $A^\circ \subseteq A$ and

2) $A \subseteq \bar{A}$

Thm 6.4.

Let $Y$ be a subspace of $X$ and $A$ be a subset of $Y$. Let $\bar{A}$ denotes the closure of $A$ in $X$. Then the closure of $A$ in $Y$ equals $\bar{A} \cap Y$.

Pf:

Given $Y$ be a subspace of $X$.
$A \subset Y$ and $\bar{A}$ is the closure of $A$ in $X$.
Let $B$ denote the closure of $A$ in $Y$.
Claim $B = \bar{A} \cap Y$.
For that claim, $B \subset \bar{A} \cap Y$ and $\bar{A} \cap Y \subset B$

W.K.T,

$\bar{A}$ is closed set in $X$.
$\Rightarrow \bar{A} \cap Y$ is a closed set in $Y$.
$\therefore \bar{A} \cap Y$ is a closed set in $Y$ containing $A$. $\rightarrow (1)$

But $B$ is a closure of $A$ in $Y$.
$\Rightarrow B$ is the smallest closed in $Y$ containing $A \rightarrow (2)$

(1) and (2), $B \subset \bar{A} \cap Y \rightarrow (*)$

By assumption,

$B$ is the closure of $A$ in $Y$.
$\therefore B$ is closed in $Y$.
$\Rightarrow B = C \cap Y$. Where $C$ is some closed set in $X$. [By thm 6.2]

Again $B$ is the closure of $A$ in $Y$.
$\Rightarrow A \subset B$
$\Rightarrow A \subset C \cap Y$
$\Rightarrow A \subset C$ and $A \subset Y$.
$\therefore C$ is a closed set in $X$ containing $A \rightarrow (3)$
But $\bar{A}$ is closure of $A$ in $X$.
$\Rightarrow \bar{A}$ is the smallest closed set in $X$ containing $A. \rightarrow (4)$.

$(3) + (4) \Rightarrow \bar{A} \subset C$
$\Rightarrow \bar{A} \cap Y \subset C \cap Y$.
$\Rightarrow \bar{A} \cap Y \subset B \rightarrow (**)$.

From $(*)$ & $(**)$ we have

$B = \bar{A} \cap Y$.

Thm 6.5.



Let A be a subset of the topological space $x$.

a, Then $x \in \bar{A} \Leftrightarrow$ Every open set $U$ containing $x$ intersect A.

b, Supposing the topological of $x$ is given by basis. then $x \in \bar{A} \Leftrightarrow$ Every basis element $B$ containing $x$ intersect A.

pf:

Statement (a) is of the form $P \Leftrightarrow Q$.

Let us prove,

ie, not $P \Leftrightarrow$ not $Q$.

So first assume $x \notin \bar{A}$.

To prove, there exist an open set $U$ containing $x$ not intersecting A.

Now, $x \notin \bar{A} \Rightarrow x \in X - \bar{A}$

$\bar{A}$ is closed set $\Rightarrow X - \bar{A}$ is open set

$\bar{A} \cap (X - \bar{A}) = \phi$

$A \cap (X - \bar{A}) = \phi$

Moreover, $A \cap (X - \bar{A}) = \phi$

$\Rightarrow$ for an open set $U (= X - \bar{A})$ not intersecting A.

$\therefore$ not $P \Rightarrow$ not $Q$.

Conversely,

Let us assume that, there exist an open set $U$ containing $x$ not intersecting A.

ie, there exist an open set $U$ containing $x$

$U \cap A = \phi$

Now, $U \cap A = \phi \Rightarrow A \subset (X - U)$

$\Rightarrow (X - U)$ is closed set containing A.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subset (X - U) \longrightarrow *$

Again $x \in U \Rightarrow x \notin (X - U)$

$\Rightarrow x \notin \bar{A}$ (by $*$)

not $Q \Leftrightarrow$ not $P$.

b) Let $x \in \bar{A}$ and $B$ be a basis element.

Claim : $B \cap A \neq \phi$

W.K.T,

Every basis element is open set.

$\therefore B$ is an open set containing $x$.

Then by part (a) $B \cap A \neq \phi$

Conversely,

Assume every basis element $B$ containing $x$ intersect $A$.

ie, $B \cap A \neq \phi$, for every $x \in B \rightarrow (**)$

Claim, $x \in \bar{A}$

By part (a), it is enough to prove for every open set $U$ containing $x$ intersecting $A$.

Let $U$ be a open set containing $x$.

basis By defn of open set, there exist a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

By assumption, $(**)$ $B \cap A \neq \phi$

$\Rightarrow U \cap A \neq \phi$

$\Rightarrow x \in \bar{A}$ [By part (a)]

$\therefore x \in \bar{A} \Leftrightarrow$ Every basis element $B$. containing $x$ intersects $A$.

Note :-

We can shorter the statement. "$U$ is an open set containing $x$" to the set" $U$ is a neighbourhood of $x$".

ie, ~~lee~~ A open set containing $x$ is called a neighbourhood of $x$.

ie, If $U$ is a neighbourhood of $x$. Then $x$ is a limit points of $A$. if $A - \{x\} \cap U \neq \phi$.

Limit point.

Defn:-

Let $X$ be a topological space and let $A$ be the subset of $X$. Let $x \in X$, is called the limit point (cluster point (or) accumulation point) of $A$ if every neighbourhood of $x$ intersects $A$ in some points. Other than $x$ itself.

Let $A$ be the subset of the topological space $X$. and $x \in X$. We say that $x$ is a limit point of $A$ if $A - \{x\} \cap U \neq \phi$.

Thm 6.6.

Let $A$ be a subset of the topological space $X$. Let $A'$ be the set of all limit points of $A$. Then $\bar{A} = A \cup A'$

Pf:-

First to prove $A \cup A' \subset \bar{A}$

Let $x \in A \cup A' \Rightarrow x \in A$ or $x \in A'$

case i,

If $x \in A$, then obviously $x \in \bar{A}$ $(A \subset \bar{A})$

[By defn of closure]

case ii,

If $x \in A'$

$\Rightarrow$ Every neighbourhood of $x$ intersects $A$ in some points other than $x$.

So every neighbourhood of $x$ intersect $A$.

$\Rightarrow x \in \bar{A}$ (By thm 6.5)

Either $x \in A$ (or) $x \in A'$, we have $x \in \bar{A}$.

$\therefore A \cup A' \subset \bar{A} \rightarrow (1)$

Conversely,

Let $x \in \bar{A}$

claim: $\bar{A} \subset A \cup A'$

case i,

If $x \in A$.

then obviously $x \in A \cup A'$

$\therefore \bar{A} \subset A \cup A'$

Case ii)

If $x \notin A$.

$\therefore$ Every neighbourhood of $x$ intersect $A$ in some points other than $x$.

$\Rightarrow x \in A'$

$\Rightarrow x \in A \cup A'$

$\therefore \bar{A} \subset A \cup A'$

Thus in both cases, $\bar{A} \subset A \cup A' \rightarrow (2)$

From (1) & (2)

$\bar{A} = A \cup A'$.

Defn:

Hausdroff space

A topological space $X$ is called a Hausdroff space. if for each pair $x_1, x_2$ of distinct points of $X$. there exist neighbourhood $U_1$ and $U_2$ of $x_1$ and $x_2$ respectively that are disjoint.

Remark:

A topological space $X$ is a Hausdroff space if every pair of distinct points of $X$ are seperated by two distinct neighbourhood.

Thm 6.8

Every finite set in a Hausdroff space $X$ is closed.

pf:

Let $X$ be a hausdroff space and $A$ be a finite set.

Claim: $A$ is closed.

Since $A$ is finite, it is a finite union of singleton set.

So in order to prove, $A$ is closed It is enough to prove $\{x_0\}$ is closed.

[$\because$ finite union of closed sets is closed]

For that prove $\{x_0\}$ contains all its limit points

[A set is said to be closed f contains all its limit points].

Let $x \neq x_0$.

Since $X$ is Housdroff space.

$\Rightarrow$ there exist a neighbourhood $U$ of $x$. a neighbourhood $V$ of $x_0$ such that $U \cap V = \phi$.

$\Rightarrow U \cap \{x_0\} = \phi$ ($\because$ By taking $V = \{x_0\}$) (32)

$\Rightarrow$ there exist a neighbourhood $U$ of $x$ not intersecting $\{x_0\}$.

$\Rightarrow x_0$ is the only limit point of $\{x_0\}$.

$\Rightarrow \{x_0\}$ Contains all its limit points.

$\Rightarrow \{x_0\}$ is closed.

$\therefore A$ is closed.

Theorem: 6.9

Let $X$ be a Hausdorff space and $A$ be a subset of $X$. then the point $x$ is the limit point of $A$ iff every neighbourhood $U$ of $x$ Containing infinitely many points of $A$.

Pf:-

Assume that, every neighbourhood $U$ of $x$ Contains infinitely many points of $A$.

Prove, that

$x$ is the limit point of $A$.

By the assumption, $U$ intersect $A$ in some point other than $x$.

$\Rightarrow x$ is a limit point of $A$.

Conversely,

Let $x$ be a limit point of $A$.

and let $U$ be an arbitrary neighbourhood of $x$.

Claim:

$U$ Contains infinitely many points of $A$.

Suppose $U$ contains only finite number of elements of $A$.

Then $U \cap A$ is finite in number.

$U \cap \{A - \{x\}\}$ is also finite.

Let the intersects be $x_1, x_2 \ldots x_m$

ie, $U \cap \{A - \{x\}\} = \{x_1, x_2 \ldots x_m\} \rightarrow (1)$

W.K.T, A finite set in $T_2$ space (hausdorff space) is closed.

$\therefore$ Set of all $\{x_1, x_2 \dots x_m\}$ is closed in $x$.

$\Rightarrow$ $X - \{x_1, x_2 \dots x_m\}$ is open in $X$.

Also consider the set,

$U \cap [X - \{x_1, x_2 \dots x_m\}]$, $U$ is open and

$X - \{x_1, x_2 \dots x_m\}$ is open in $x$.

$\Rightarrow U \cap [X - \{x_1, x_2 \dots x_m\}]$ is also open in $x \rightarrow (2)$

Also,

$x \in U$ and $x \in X - \{x_1, x_2 \dots x_m\}$.

$\Rightarrow x \in U \cap [X - \{x_1, x_2 \dots x_m\}] \rightarrow (3)$

by (2) & (3)

$U \cap [X - \{x_1, x_2 \dots x_m\}]$ is open set contain $x$ but not intersecting $A - \{x\}$ [by (1)]

$\Rightarrow$ there exist a neighbourhood of $x$ which does not intersect $A$ in some point other than $x$.

$\Rightarrow x$ is not a limit point of $A$.

Which is contradiction.

$\therefore$ $U$ containing infinitely many points of $A$.

Thm: 6.10.

i) Every simply ordered set is Hausdroff space in the ordered topology.

ii) The product of two hausdroff space is Hausdroff space.

iii) A subspace of a hausdroff space is a hausdroff space.

Pf:

i) claim,

Every simply ordered set is a Hausdroff space.

let $X$ be simply ordered set.

claim,

$X$ is a Hausdroff space.

let $x, y \in X$ such that $x \neq y$.

$X$ is a well ordered set.

$\Rightarrow x < y$ (or) $y < x$.

Assume $x < y$,

Consider the element $a, b, c, d$ in $X$ such that $a < x < b$, $c < y < d$ with $(a, b) \cap (c, d) = \phi$

$x \in (a,b)$ and $(a,b)$ is open in $X$.
$\Rightarrow (a,b)$ is neighbourhood of $x$.
$y \in (c,d)$ and $(c,d)$ is open in $X$.
$\Rightarrow (c,d)$ is neighbourhood of $y$.
$\therefore (a,b)$ and $(c,d)$ are disjoint neighbourhood of $x$ and $y$ respectively.
$\therefore X$ is a Housdroff space.

ii)

Given $X$ and $Y$ are Housdroff space.
Let $x_1 \times y_1, x_2 \times y_2 \in X \times Y$ such that $x_1 \times y_1 \neq x_2 \times y_2$
$x_1 \times y_1 \neq x_2 \times y_2 \Rightarrow x_1 \neq x_2$ (or)
$x_1 = x_2$ and $y_1 \neq y_2$
Since $x_1 \times y_1$ and $x_2 \times y_2 \in X \times Y$.
$\Rightarrow x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$.
$x_1 \neq x_2$ in $X$ and $X$ is a hausdroff space.
$\Rightarrow$ there exists a neighborhood $U_1$ and $U_2$ of $x_1, x_2$ of $X$ such that $U_1 \cap U_2 = \phi \rightarrow (1)$

$\|^{ly}$
$y_1 \neq y_2$ in $Y$ and $Y$ is a haudroff space.
$\Rightarrow$ there exists a neighbourhood $V_1$ and $V_2$ of $y_1, y_2$ of $Y$ such that $V_1 \cap V_2 = \phi$
Consider the open sets $U_1 \times V_1$ and $U_2 \times V_2$ in $X \times Y$.
Now,
$x_1 \times y_1 \in U_1 \times V_1$ and $x_2 \times y_2 \in U_2 \times V_2$ are neighbourhood of $x_1 \times y_1$ and $x_2 \times y_2$ respectively.
Also, $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$
$= \phi \times \phi = \phi$
Also, $(U_1 \times V_1)$ and $(U_2 \times V_2)$ are disjoint neighbourhood of $x_1 \times y_1$ and $x_2 \times y_2$ ~~respectively~~ in $X \times Y$.
$\therefore X \times Y$ is also a hausdroff space.

iii)

Let $X$ be a hausdroff space and $Y$ be a subspace of $X$.

Claim, Y is also Hausdroff space. (35)

let $y_1 \neq y_2 \in Y$.

Claim,

there exists two disjoint open sets containing $y_1$ and $y_2$ respectively in Y.

Since Y is a subspace of X.

$y_1, y_2 \in Y \Rightarrow y_1, y_2 \in X$.

Since X is a hausdroff space.

these two distinct points $y_1$ and $y_2$ are seperated by two disjoint open sets U and V in X which are containing $y_1$ and $y_2$ respectively.

$U \cap V = \phi$

Where U is open set in X containing $y_1$ and V is open set in X containing $y_2$.

ie, $y_1 \in U$, $y_2 \in V$ and $U \cap V = \phi$

Clearly,

$y_1 \in U \cap Y$ and $y_2 \in V \cap Y$.

$U \cap Y$ and $V \cap Y$ are open in Y.

[$\because$ Y is subspace of X]

$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \phi \cap Y = \phi$

$\therefore y_1 \neq y_2$ in Y have a disjoint neighbourhoods, $U \cap Y$ and $V \cap Y$ respectively.

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(1).

## UNIT - 2

## CONTINUOUS FUNCTIONS

Defn:

Note:

$f^{-1}(V) = \{x \in X / f(x) \in V\}$

Let the topological space Y is given with the basis $\mathcal{B}$. then $V = \bigcup_{\alpha \in J} B_\alpha$ Where $B_\alpha \in \mathcal{B}$.

In order to prove $f^{-1}(V)$ is open in X.

To prove,

$f^{-1}(B_\alpha)$ is open in X [ for the constant function ]

$\|^{by}$, interms of subbasis element of one $B = S_1 \cap S_2 \cap \dots \cap S_n$.

# Topology

## UNIT - 2

Unit-II

continuous functions - the product topology
the metric topology - the quotient topology.

Unit-III

Claim, $Y$ is also Hausdroff space. (35)

let $y_1 \neq y_2 \in Y$.

Claim,

there exists two disjoint open sets containing $y_1$ and $y_2$ respectively in $Y$.

Since $Y$ is a subspace of $X$.

$y_1, y_2 \in Y \Rightarrow y_1, y_2 \in X$.

Since $X$ is a hausdroff space.

these two distinct points $y_1$ and $y_2$ are seperated by two disjoint open sets $U$ and $V$ in $X$ which are containing $y_1$ and $y_2$ respectively.

$U \cap V = \phi$

Where $U$ is open set in $X$ containing $y_1$ and $V$ is open set in $X$ containing $y_2$.

ie, $y_1 \in U$, $y_2 \in V$ and $U \cap V = \phi$

Clearly,

$y_1 \in U \cap Y$ and $y_2 \in V \cap Y$.

$U \cap Y$ and $V \cap Y$ are open in $Y$.

[$\because Y$ is subspace of $X$]

$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \phi \cap Y = \phi$

$\therefore y_1 \neq y_2$ in $Y$ have a disjoint neighbourhoods, $U \cap Y$ and $V \cap Y$ respectively.

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(1).

# UNIT - 2

## CONTINUOUS FUNCTIONS

Defn:

Note:

$f^{-1}(V) = \{x \in X / f(x) \in V\}$

Let the topological space $Y$ is given with the basis $\mathcal{B}$. then $V = \bigcup_{\alpha \in J} B_\alpha$ where $B_\alpha \in \mathcal{B}$.

In order to prove $f^{-1}(V)$ is open in $X$.

To prove,

$f^{-1}(B_\alpha)$ is open in $X$ [for the constant function]

$\|^{ly}$, interms of subbasis element of one $B = S_1 \cap S_2 \cap \dots \cap S_n$.

then $f^{-1}(B)$ is open. (2)

$\Rightarrow f^{-1}(S_i)$ is open in $X$. for every $i = 1$ to $n$.

Since $f^{-1}(B) = f^{-1}(S_1) \cap f^{-1}(S_2) \cap \dots \cap f^{-1}(S_n)$.

Thm 7.1

Let $X$ and $Y$ be topological space. Let $f: X \to Y$ then the following are equivalent.

i) $f$ is continuous.

ii) Every subset $A$ of $X$ one has $f(\bar{A}) \subseteq \overline{f(A)}$

iii) for every closed set $B$ of $Y$ the set $f^{-1}(B)$ is closed in $X$.

iv) for each $x \in X$ and each neighbourhood $V$ of $f(x)$ there exist a neighbourhood $U$ of $x$ such that $f(U)$ is subset of $V$.

Pf:

(i) $\Rightarrow$ (ii)

Let $f: X \to Y$ be continuous.

Claim: $f(\bar{A}) \subseteq \overline{f(A)}$ where $A$ is subset of $X$.

Let $x \in \bar{A} \Rightarrow f(x) \in f(\bar{A})$

Let $V$ be a neighbourhood of $f(x)$ in $Y$.

Now, $V$ is open in $Y$ and $f: X \to Y$ is continuous.

$\Rightarrow f^{-1}(V)$ is open in $X \to (1)$

Also, $f(x) \in V \Rightarrow x \in f^{-1}(V) \to (2)$

(1) & (2) $\Rightarrow f^{-1}(V)$ is an open in $X$ containing $x$ but $x \in \bar{A}$

Again $x \in \bar{A} \Rightarrow$ Every neighbourhood of $x$ intersect $A$.

In particular, (By thm 6.5)

$f^{-1}(V)$ intersects $A$.

$\therefore f^{-1}(V) \cap A \neq \phi$ (by (3))

Let $y \in f^{-1}(V) \cap A \Rightarrow y \in f^{-1}(V)$ and $y \in A$.

$\Rightarrow f(y) \in V$ and $f(y) \in f(A)$

$\Rightarrow V \cap f(A) \neq \phi$.

Since $V$ is an arbitrary neighbourhood of $f(x)$ and $V$ intersects $f(A)$.

$\Rightarrow$ Every neighbourhood of $f(x)$ intersect $f(A)$.

$\Rightarrow f(x) \in \overline{f(A)}$ (by thm 6.5)

$\therefore f(\bar{A}) \subseteq \overline{f(A)}$.

(ii) $\Rightarrow$ (iii) (3)

Let $A$ be a subset of $X$ and $f(\bar{A}) \subset \overline{f(A)}$.

Claim: If $B$ is closed in $Y$ then $f^{-1}(B)$ is closed in $X$.

Let $A = f^{-1}(B)$.

Claim, $A$ is closed in $X$

For that claim $A = \bar{A}$

Obviously, $A \subset \bar{A}$

So, to prove $\bar{A} \subset A$.

Let $x \in \bar{A}$

$\Rightarrow f(x) \in f(\bar{A}) \subset \overline{f(A)}$

$\Rightarrow f(x) \in \overline{f(A)}$

$\Rightarrow f(x) \in \bar{B}$ $\quad [\because f^{-1}(B) = A \Rightarrow B = f(A)]$

$\Rightarrow f(x) \in B$ $\quad$ [By hypothesis $B$ is closed in $Y$]

$\Rightarrow x \in f^{-1}(B)$

$x \in A$

$x \in A$ and $\bar{A} \subset A \Rightarrow A = \bar{A}$

$\Rightarrow A$ is closed in $X$.

$\Rightarrow f^{-1}(B)$ is closed in $X$.

(iii) $\Rightarrow$ (iv)

Let $B$ is closed in $Y$.

then $f^{-1}(B)$ is closed in $X$.

claim: $f: X \to Y$ is continuous.

For that, Let $V$ be open in $Y$ and claim $f^{-1}(V)$ is open in $X$.

Given $V$ is open in $Y$.

$\Rightarrow Y - V$ is closed in $Y$.

$\Rightarrow f^{-1}(Y - V)$ is closed in $X$.

$\Rightarrow f^{-1}(Y) - f^{-1}(V)$ is closed in $X$.

$\Rightarrow X - f^{-1}(V)$ is closed in $X$ $\quad [\because f^{-1}(Y) = X]$

$\Rightarrow f^{-1}(V)$ is open in $X$

$f$ is continuous.

(i) $\Rightarrow$ (iv)

Let $f$ be continuous.

Claim, for every $x \in X$ and for every neighbourhood $V$ of $f(x)$ there exist a neighbourhood $U$ of $x$ such that $f(U) \subset V$.

(4) Let $V$ be a neighbourhood of $f(x)$.

$\Rightarrow f(x) \in V$

$\Rightarrow x \in f^{-1}(V)$.

Since $V$ is open in $Y$ and $f$ is continuous. We have,

$f^{-1}(V)$ is open set in $X$.

Now, $x \in f^{-1}(V)$ and $f^{-1}(V)$ is open in $X$.

$\Rightarrow$ there exist a neighbourhood $U$ of $x$ such that $x \in U \subset f^{-1}(V)$.

$\Rightarrow f(U) \subset V$.

$\therefore$ there exist a neighbourhood of $x$ such that $f(U) \subset V$.

(iv) $\Rightarrow$ (i)

Let for every $x \in X$ for every neighbourhood $V$ of $f(x)$.

there is a neighbourhood $U$ of $x$ such that $f(U) \subset V$.

Claim: $f: X \to Y$ is continuous

Let $V$ be an open set in $Y$.

Claim, $f^{-1}(V)$ is open in $X$.

Let $x \in f^{-1}(V) \Rightarrow f(x) \in V$.

By hypothesis,

there exist a neighbourhood $U$ of $x$ such that $f(U) \subset V$,

$\Rightarrow U \subset f^{-1}(V)$.

$\therefore$ We have $x \in U \subset f^{-1}(V)$.

there exist a neighbourhood of $x$ such that,

$x \in U \subset f^{-1}(V)$.

$\Rightarrow f^{-1}(V)$ is open in $X$.

Thus $V$ is open in $Y$.

$\Rightarrow f^{-1}(V)$ is open in $X$.

$\therefore f$ is continuous.

Defn:

Open Mapping.

Let $f: X \to Y$ where $X$ and $Y$ are topological spaces. then the function $f$ is said to be open if $U$ is open in $X$. $\Rightarrow f(U)$ is open in $X$.

Thm 7.2

Rules for constructing continuous function.

Let $X, Y$ and $Z$ be topological spaces.

a, (Constant function) If $f: X \to Y$ maps all of $X$ into the single point $y_0$ of $Y$. then $f$ is continuous.

b, (Inclusion) If $A$ is a subspace of $X$, the inclusion function $j: A \to X$ is continuous.

c) (Composites): If $f: X \to Y$ and $g: Y \to Z$ are continuous, then the map $g \circ f: X \to Z$ is continuous.

d, (Restricting the domain): If $f: X \to Y$ is continuous and if $A$ is a subspace of $X$, then the restricted function $f/A: A \to Y$ is continuous.

e, (Restricting or expanding the range) Let $f: X \to Y$ be continuous. If $Z$ is a subspace of $Y$ containing the image set $f(X)$. then the function $g: X \to Z$ obtained by restricting the range of $f$ is continuous. If $Z$ is a space having $Y$ as a subspace.

then the function $h: X \to Z$ obtained by expanding the range of $f$ is continuous.

f, (Local formulation of continuity):

The map $f: X \to Y$ is continuous if $X$ can be written as the union of open sets $U_\alpha$ such that $f|U_\alpha$ is continuous for each $\alpha$.

g, Continuity at each points:

The map $f: X \to Y$ is continuous if for each $x \in X$ and for each neighbourhood of $f(x)$ there exist a neighbourhood $U$ of $x$ such that $f(U) \subset V$.

Pf:

a,

Let $f: X \to Y$ be a continuous function.

Such that $f(x) = y_0 \;\forall\; x \in X$.

Let $V$ be an open subset of $Y$.

Claim that ; $f^{-1}(V)$ is open in $X$.

If $y_0 \in V$.

then $f^{-1}(V) = X$, $X$ is open in $X$.

If $y_0 \notin V$. then $f^{-1}(V) = \phi$, $\phi$ is open in $X$.

$\therefore f^{-1}(V)$ is open in $X$.

$\therefore f$ is continuous.

b, Let $j : A \to X$ be a inclusion function and $A$ is a subspace of $X$.

Claim,

$j$ is continuous.

Let $U$ be an open subset of $X$.

Then by defn. of inclusion for $j^{-1}(U) = U \cap A$.

$U$ is open in $X \Rightarrow U \cap A$ is open in $A$.

[by defn of subspace]

$\therefore U$ is open in $X \Rightarrow$ ~~$f^{-1}(U)$~~ $j^{-1}(U)$ is open in $A$.

$j : A \to X$ is continuous.

c, Given $f : X \to Y$ and $g : Y \to Z$ are continuous.

Claim, $g \circ f : X \to Z$ is continuous.

Let $V$ be any open subset of $Z$.

$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \to (1)$

Now, $g : Y \to Z$ is continuous.

$\therefore V$ is open in $Z \Rightarrow g^{-1}(V)$ is open in $Y$.

$f : X \to Y$ is continuous.

$\therefore g^{-1}(V)$ is open in $Y \Rightarrow f^{-1}(g^{-1}(V))$ is open in $X$.

$\therefore g \circ f$ is continuous.

$\therefore f$ and $g$ are continuous

$\Rightarrow g \circ f$ is also continuous.

d, Let $f : X \to Y$ continuous.

Let $A$ be a subspace of $X$.

Let $f/_A : A \to Y$ be the restriction function.

Claim : $f|_A$ is continuous.

Let $j : A \to X$ be an inclusion map.

Clearly, $f|_A = f \circ j$

WKT, The inclusion function is continuous.

$\Rightarrow j$ is continuous.

$\Rightarrow f \circ j$ is continuous. [$\because f$ and $j$ are continuous]

$\therefore f|_A$ is continuous.

Case i)

$f: X \to Y$ is continuous also $f(x) \subset Z \subset Y$.

claim: $g: X \to Z$ is continuous.

For that claim, $B$ is open set in $Z$.

$\Rightarrow g^{-1}(B)$ is an open set in $X$.

Let $B$ be the open in $Z$.

Then $B = U \cap Z$. Where $U$ is open in $Y$.

Consider,

$$g^{-1}(B) = g^{-1}(U \cap Z) = g^{-1}(U) \cap g^{-1}(Z)$$

$$= f^{-1}(U) \cap X \qquad [\because f(x) \subseteq Z] \quad X \subseteq f^{-1}(Z)$$

$$= f^{-1}(U)$$

But $f$ is continuous and $U$ is open in $Y$.

$\Rightarrow f^{-1}(U)$ is open in $X$.

$\Rightarrow g^{-1}(B)$ is open in $X$.

$\Rightarrow g$ is continuous.

Case ii)

Let $f: X \to Y$ be continuous.

Let $Z$ be a space having $Y$ as a subspace.

The map $h: X \to Z$ is obtained by expanding the range of $f$.

Claim: $h: X \to Z$ is continuous.

Let $j: Y \to Z$ be inclusion map.

We have $h = j \circ f$ [ie, $j \circ f: X \to Z$]

WKT, the inclusion map $j$ is continuous.

$\therefore j \circ f$ is continuous.

$\Rightarrow h$ is continuous.

Given $X = \bigcup_\alpha U_\alpha$ where each $U_\alpha$ is open in $X$.

Also, given $f|_{U_\alpha}: U_\alpha \to Y$ is continuous for each $\alpha$.

Claim,

$f: X \to Y$ is continuous.

Let $V$ be open set in $Y$.

claim: $f^{-1}(V)$ is open in $X$.

W.K.T, $f^{-1}(V) \cap U_\alpha = (f|U_\alpha)^{-1}(V) \rightarrow (1)$

$f^{-1}(V) = f^{-1}(V) \cap X$ (∵ by set notation)

$= f^{-1}(V) \cap \bigcup_\alpha U_\alpha = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha)$

$f^{-1}(V) = \bigcup_\alpha (f/U_\alpha)^{-1}(V)$ [by (1)].

But $f|U_\alpha : U_\alpha \rightarrow Y$ is continuous for each $\alpha$.

∴ $(f/U_\alpha)^{-1}(V)$ is open in $U_\alpha$.

But $U_\alpha$ is open in $X$.

$\Rightarrow (f|U_\alpha)^{-1}(V)$ is open in $X$.

W.K.T,

arbitrary union of open sets is open.

$\bigcup_\alpha (f/U_\alpha)^{-1}(V)$ is open in $X$.

$f^{-1}(V)$ is open in $X$.

∴ $f$ is continuous.

9, Let $f: X \rightarrow Y$

Assume that, for each $x \in X$ and each neighbourhood of $f(x)$, there exists a neighbourhood $U$ of $x$ such that $f(U) \subset V$.

Claim, $f$ is continuous.

Let $V$ be any open set in $Y$.

Consider, $f^{-1}(V)$ and let $x \in f^{-1}(V)$ be arbitrary.

$f(x) \in V$.

$V$ is open, since $V$ is neighbourhood of $f(x)$.

By hypothesis, there exist neighbourhood $U_x$ of $x$ such that,

$f(U_\alpha) \subset V$.

$\therefore U_\alpha \subset f^{-1}(V)$

$\Rightarrow \bigcup_\alpha U_\alpha = f^{-1}(V)$

Since each $U_\alpha$ is open in $X$.

$\Rightarrow \bigcup_\alpha U_\alpha$ is open in $X$.

$\Rightarrow f^{-1}(V)$ is open in $X$.

∴ $f$ is continuous all the point $x$.

$f$ is continuous function.

(9)

Theorem :

Pasting Lemma.

Let $X = A \cup B$ where $A$ and $B$ are closed in $X$. Let $f: A \to Y$ and $g: B \to Y$ be continuous function. If $f(x) = g(x)$ $\forall x \in A \cap B$, then $f$ and $g$ combine to give a continuous function $h: X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Pf:

Given $X = A \cup B$, where $A$ and $B$ are closed in $X$. also given $f: A \to Y$ be continuous and $g: B \to Y$ be continuous. such that

$$f(x) = g(x) \quad \forall \quad x \in A \cap B.$$

Claim,

$h: X \to Y$ defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is continuous.

Let $C$ be closed set in $Y$.

Claim: $h^{-1}(C)$ is closed in $X$.

Since $h^{-1}(C) \subset X$.

We have $h^{-1}(C) = h^{-1}(C) \cap X$

$$= h^{-1}(C) \cap (A \cup B) \quad [\because X = A \cup B]$$

$$= (h^{-1}(C) \cap A) \cup (h^{-1}(C) \cap B)$$

$$= f^{-1}(C) \cup g^{-1}(C) \longrightarrow (1)$$

$[\because f^{-1}(C) \subseteq A$ $\quad h(x) = f(x)$ if $x \in A$

$g^{-1}(C) \subseteq B]$ $\quad h(x) = g(x)$ if $x \in B$.

Now, $f: A \to Y$ be continuous and $C$ be closed in $Y$.

$\Rightarrow f^{-1}(C)$ is closed in $A$.

$||^{ly}$,

$g: B \to Y$ be continuous and $C$ is closed in $Y$.

$\Rightarrow g^{-1}(C)$ is closed in $B$.

Also, $g^{-1}(C)$ is closed in $B$ and $B$ is closed in $X$.

$\Rightarrow g^{-1}(C)$ is closed in $X$.

Similarly,

$f^{-1}(C)$ is also closed in $X$.

$\therefore f^{-1}(C)$ and $g^{-1}(C)$ both closed in $X$.

$\Rightarrow f^{-1}(C) \cup g^{-1}(C)$ is closed in $X$.

$\Rightarrow h^{-1}(C)$ is closed in $X$ [by (1)]
Thus $C$ is closed in $Y$.
$\therefore h^{-1}(C)$ is closed in $X$.
$h$ is continuous.

---

Thm :-

Maps into product

Let $f: A \to X \times Y$ be a given function defined by $f(a) = (f_1(a), f_2(a))$. then $f$ is continuous iff the function $f_1 : A \to X$ ; $f_2 : A \to Y$ are continuous.

[This $f_1$ and $f_2$ are called co-ordinate funcl. of $f$.]

pf:-

Let $f: A \to X \times Y$ is a continuous function defined by, $f(a) = (f_1(a), f_2(a))$.
Where
$f_1 : A \to X$ and $f_2 : A \to Y$
Claim: $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.
Consider the projection for, $\pi_1 : X \times Y \to X$. defined by $\pi_1(x, y) = x$.
Claim, $\pi_1$ is continuous.
Let $U$ is an open set in $X$.
then, $\pi_1^{-1}(U) = U \times Y$.
Now, $U$ is open in $X$ and $Y$ is open in $Y$.
$\Rightarrow U \times Y$ is open in $X \times Y$
$\Rightarrow \pi_1^{-1}(U)$ is open in $X \times Y$. [Under the product topology]
$\Rightarrow \pi_1$ is continuous.
Similarly,
We can prove that the projection for $\pi_2 : X \times Y \to Y$ defined by $\pi_2(x, y) = y$ is also continuous.
Let $V$ is an open set in $Y$ and $X$ is an open set in $X$.
$\pi_2^{-1}(V) = X \times V$
[Under the product topology]
$\Rightarrow X \times V$ is open in $X \times Y$.
$\Rightarrow \pi_2^{-1}(V)$ is open in $X \times Y$.
$\Rightarrow \pi_2$ is continuous.

(11) Thus, we have proved $\pi_1$ and $\pi_2$ are continuous.

Given,

$f: A \to X \times Y$ is continuous.

Consider, the function $(\pi_1 \circ f): A \to X$ $[A \to X \times Y \to X]$

Moreover,

$(\pi_1 \circ f)(a) = \pi_1(f(a))$ where $a \in A$.

$= \pi_1(f_1(a), f_2(a))$ [by defn of $f$]

$= f_1(a)$. [by defn of $\pi_1$]

Since $f$ and $\pi_1$ are continuous.

$\Rightarrow (\pi_1 \circ f)$ is continuous [$\because$ composition of two cont. function is cont.]

$\Rightarrow f_1$ is continuous.

$\|^{by}$,

Consider, the function $(\pi_2 \circ f): A \to Y$ $[A \to X \times Y \to Y]$

Moreover

$(\pi_2 \circ f)(a) = \pi_2(f(a)) \; \forall \; a \in A$

$= \pi_2(f_1(a), f_2(a))$ [Defn. of $f$]

$= f_2(a)$ (Defn. of $\pi_2$)

Now, $\pi_2$ and $f$ is continuous.

$\Rightarrow (\pi_2 \circ f)$ is continuous.

$\Rightarrow f_2$ is continuous.

Thus, $f$ is continuous $\Rightarrow$ both $f_1$ and $f_2$ are continuous.

Conversely,

Assume that, let $f_1: A \to X$ and $f_2: A \to Y$ both continuous.

Claim: $f: A \to X \times Y$ is continuous.

Let $U \times V$ be an open set in $X \times Y$ under the product topology.

Claim: $f^{-1}(U \times V)$ is open in $A$.

For that, first claim $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$.

Let $a \in f^{-1}(U \times V)$

$\Leftrightarrow f(a) \in U \times V$

$\Leftrightarrow (f_1(a), f_2(a)) \in U \times V$.

$\Leftrightarrow f_1(a) \in U$ and $f_2(a) \in V$

$\Leftrightarrow a \in f_1^{-1}(U)$ and $a \in f_2^{-1}(V)$.

$\Leftrightarrow a \in f_1^{-1}(U) \cap f_2^{-1}(V)$.

$\therefore f_1^{-1}(U\times V) = f_1^{-1}(U) \cap f_2^{-1}(V) \longrightarrow (1)$ (12)

Now, $U$ is open in $X$ and $f_1 : A \to X$ is Continuous.

$\Rightarrow f_1^{-1}(U)$ is open in $A$.

Similarly,

$V$ is open in $Y$ and $f_2 : A \to Y$ is continuous.

$\Rightarrow f_2^{-1}(V)$ is open in $A$.

$\Rightarrow f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in $A$.

$\Rightarrow f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in $A$.

$\Rightarrow f_1^{-1}(U\times V)$ is open in $A$ [by (1)].

Thus $U\times V$ is open in $X\times Y$

$f^{-1}(U\times V)$ is open in $A$.

$f$ is Continuous.

---

Result :

Let $f : X \to Y$ be bijection. Then $f$ is a homeomorphism iff $f(U)$ is open in $Y$ iff $U$ is open in $X$.

Pf :-

$\Rightarrow$

Let $f$ be a homeomorphism.

Claim : $U$ is open in $X$ iff $f(U)$ is open in $Y$.

Let $U$ is open in $X$.

Let $f$ is homeomorphism iff $f$ and $f^{-1}$ both are Continuous.

Consider, $f^{-1} : Y \to X$)

Since $f^{-1}$ is continuous and $U$ is open in $X$ iff $(f^{-1})^{-1}(U)$ is open in $Y$ if $f(U)$ is open in $Y$.

Conversely,

$U$ is open in $X$ iff $f(U)$ is open in $Y$.

Claim : $f$ is a homeomorphism.

Given $f$ is a bijection.

ie, to claim, $f$ and $f^{-1}$ are Continuous.

first Claim, $f$ is Continuous.

Where $f : X \to Y$

Let $V$ be an open in $Y$.

Then $V = f(U)$ for some $U$ is in $X$.

$V$ is open in $Y \Rightarrow f(U)$ is open in $Y$
$\Rightarrow U$ is open in $X$.

Then $V$ is open in $Y \Rightarrow f^{-1}(V)$ is open
$f$ is Continuous.

Next claim,
$f^{-1}: Y \to X$ is continuous.
Let $U$ be open in $X$.

Claim : $(f^{-1})^{-1}(U)$ is open in $Y$.

W.K.T $(f^{-1})^{-1}(U) = f(U)$
Since $U$ is open in $X$.
$\Rightarrow f(U)$ is open in $Y$ (by given)
$\therefore (f^{-1})^{-1}(U)$ is open in $Y$.
$\Rightarrow f^{-1}$ is continuous.
$\therefore f$ is homeomorphism.

---

Note:-

If $f$ is a homeomorphism between $X$ and $Y$ so it steps up corresponds between in element of a topological space $X$ and elements of topological space $Y$.

$\therefore$ Any property of $X$ which can be entirely described in term of topology $X$ is also true for $Y$ provided $X$ and $Y$ are homeomorphism.

---

Product topology on $\underset{\alpha \in J}{\times} X_\alpha$.

Defn :-

In the earlier section, we define the product topology on the product space $X \times Y$. Where $X$ and $Y$ are topological space. Now generalised this definition to arbitrary Cartesian product.

There are two ways of generating the open sets in $\times X_\alpha$ which yields two types at topology that are box topology and product topology in $\underset{\alpha \in J}{\times} X_\alpha$.

---

Box topology :-

Defn :-

Let collection $\{X_\alpha\}_{\alpha \in J}$ be an index family of topological spaces. Consider the Cartesian product

$\prod_{\alpha \in J} X_\alpha$. The basis $\mathcal{B}$ for a topology on $\prod_{\alpha \in J} X_\alpha$ be the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where $U_\alpha$ is open in $X_\alpha$, for each $\alpha \in J$.

ie, $\mathcal{B} = \{ \prod_{\alpha \in J} U_\alpha / U_\alpha$ is open in $X_\alpha$, Verify $\}$

the topology generated by a basis called the box box topology.

Product topology on $\prod_{\alpha} X_\alpha$. [Interior of subbasis]

Defn:-

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological space. Let $\prod_{\alpha \in J} X_\alpha$ be the Cartesian product of $X_\alpha$'s. For each $\beta \in J$. Consider the projection mapping $\pi_\beta$ defined as $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ by $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$.

ie, The function $\pi_\beta$ is assigning each element of the product space to its $\beta^{th}$ co-ordinate. For each $\beta \in J$. $\boxed{\text{Let}}$ $S_\beta = \{ \pi_\beta^{-1}(U_\beta) / U_\beta$ is open in $X_\beta \}$

Let $S$ denote the union of these collection
$S = \bigcup_{\beta \in J} S_\beta$.

The topology generated by the subbasis $S$ is called the product topology.

Thm 8.1

How does the product topology differ from the box topology.

Pf:-

W.K.T,

box topology on product $X_\alpha$ has the basis $\mathcal{B} = \{ \prod U_\alpha / U_\alpha$ is open in $X_\alpha \}$

The product topology on product $X_\alpha$ has the basis.

$\mathcal{B} = \{ \prod U_\alpha / U_\alpha$ is open in $X_\alpha$ and $U_\alpha = X_\alpha$ except for finitely many values of $\alpha \}$

Derivation of the basis $\mathcal{B}$ for product topology is as follows, W.K.T, $\mathcal{S} = \cup S_\beta$ Where $S_\beta = \{\pi_\beta^{-1}(U_\beta) / U_\beta$ is open in $X_\beta\}$

Subbasis for the product topology.

Let $\mathcal{B}$ be the basis generated by $\mathcal{S}$. Then $\mathcal{B}$ is the collection of all finite intersection of members of $\mathcal{S}$.

If we intersects a finite number of elements of $\mathcal{S}$ all comming from the same $S_\beta$ we denote get new membership.

$$\pi_\beta^{-1}(U_1) \cap \pi_\beta^{-1}(U_2) \cap \dots \cap \pi_\beta^{-1}(U_n).$$
$$= \pi_\beta^{-1}(U_1 \cap U_2 \cap \dots U_n)$$
$$= \pi_\beta^{-1}(V) \quad \text{Where } V = U_1 \cap \dots \cap U_n.$$

3

Since $U_1, U_2 \dots U_n$ are open in $X_\beta$.

$\Rightarrow V = U_1 \cap U_2 \cap \dots U_n$ is open in $X_\beta$.

$\therefore \pi_\beta^{-1}(V) \in S_\beta$.

We have not get any new basis members.

$\therefore$ A general basis element $B \in \mathcal{B}$, is of the form,

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

Where each $U_{\beta_i}$ open in $X_{\beta_i}$, $i = 1$ to $n$.

Let $x = (x_\alpha) \in \prod_{\alpha \in J} X_\alpha$.

$x \in B \Rightarrow x \in \pi_{\beta_i}^{-1}(U_{\beta_i}) \ \forall \ i = 1, 2 \dots n$

$\Leftrightarrow \pi_{\beta_i}(x) \subset U_{\beta_i} \ \forall \ i = 1, \dots n.$

$\Leftrightarrow \pi_{\beta_i}(x_\alpha) \subseteq U_{\beta_i} \ \forall \ i = 1, \dots n.$ $[\because x = x_\alpha]$

$\Leftrightarrow x_{\beta_i} \subset U_{\beta_i}$ [by the defn. of projection of $\pi_\beta$].

i.e, $x \in B \Leftrightarrow \beta_i^{th}$ co-ordinate of $x$ belongs to $U_{\beta_i}$ but there is no restriction for the $\alpha^{th}$ co-ordinate of $x$ iff $\alpha$ is not one of the indices $\beta_1, \beta_2 \dots \beta_n$.

$B = \prod_{\alpha \in J} U_\alpha$ Where $U_\alpha$ is open in $X_\alpha$ for $U_\alpha = X_\alpha$. each $\alpha$ and except for finitely many values of $\alpha$.

$\mathcal{B} = \{\prod_{\alpha \in J} U_\alpha / U_\alpha$ is open in $X_\alpha$ and $U_\alpha = X_\alpha$ except for a finitely many values of $\alpha\}$

Which is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

Remark:-

For finite products $\prod_{\alpha=1}^{n} X_\alpha$, the two topologies are precisely the same.

---

Metric topology:-

If $y$ is the point of the basis element $B_d(x,\varepsilon)$ then there is a basis element $B_d(y,\delta)$ such that $y \in B_d(y,\delta) \subset B_d(x,\varepsilon)$.

Pf:-

Let $\delta$ be the positive number such that $\delta = \varepsilon - d(x,y)$.

Claim

$$B_d(y,\delta) \subset B_d(x,\varepsilon)$$

Let $z \in B_d(y,\delta) \Rightarrow d(y,z) < \delta$

$$\Rightarrow d(y,z) < \varepsilon - d(x,y)$$

$$\Rightarrow d(x,y) + d(y,z) < \varepsilon.$$

$$\Rightarrow d(x,z) < \varepsilon.$$

$$\therefore B_d(y,\delta) \subset B_d(x,\varepsilon).$$

---

Ex:-

Given a set $X$, defined $d$ such that,

$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ Verify $d$ is Metric [$d$ discrete metric]

Soln:-

i, Given $d(x,y) = 1$ (or) $0$.

$$d(x,y) \geq 0 \longrightarrow (1)$$

Also given $d(x,y) = 0 \Rightarrow x = y$:

$\therefore$ The equivalently in (1) holds for $x = y$.

ii, Claim, $d(x,y) = d(y,x) \ \forall \ x,y \in X$.

Suppose $x \neq y$.

then $d(x,y) = 1$ and $d(y,x) = 1$.

Suppose $x = y$.

then, $d(x,y) = 0$ and $d(y,x) = 0$.

$\therefore d(x,y) = d(y,x) \ \forall \ x,y \in X$.

iii, Claim $d(x,z) \leq d(x,y) + d(y,z)$.

case i)

If $x \neq y \neq z$.

$d(x,z) = 1$, $d(x,y) = 1$ and $d(y,z) = 1$

$d(x,y) + d(y,z) = 1+1 = 2$.

but $d(x,z) = 1$

$d(x,z) < d(x,y) + d(y,z)$.

case ii)

a) If $x = y \neq z$.

$d(x,y) + d(y,z) = 0+1 = 1$

But $d(x,z) = 1$

$\therefore d(x,z) = d(x,y) + d(y,z)$.

ii, b) If $x \neq y = z$.

$d(x,y) + d(y,z) = 1+0 = 1$ 5

$d(x,z) = 1$

$d(x,z) = d(x,y) + d(y,z)$.

case iii)

If $x = y = z$.

$\Rightarrow d(x,y) + d(y,z) = 0+0 = 0 = d(x,z)$

$\therefore d(x,z) = d(x,y) + d(y,z)$.

In all the cases, $d(x,z) \leq d(x,y) + d(y,z)$.

$\therefore$ d is Metric.

---

2 PT the Metric $d$ ($d$-discrete metric) induces the (*) discrete topology.

Pf:-

Consider the ball $B_d(x,1) = \{ y / d(x,y) < 1 \}$

$= \{ y / d(x,y) = 0 \}$

$= \{ x \}$.

So pf $\mathcal{B}$ is the basis for the Metric topology.

Then, $\mathcal{B} = \{ B_d(x,1) / x \in X \}$

$= \{ \{x\} / x \in X \}$.

= The collection of all singleton sets.

Which is the basis for the discrete topology.

Hence Discrete Metric induces a discrete topology.

## Standard Metric

Defn:

Let $R$ be the set of all real numbers and $d: X \times X \to R$ defined by $d(x,y) = |x-y|$. This metric is called standard metric on $R$.

Result:

The standard metric $d$ induces ordered topology (or) set topology on $R$.

Pf:-

Claim, standard metric $d$ is a metric.

i) $d(x,y) = |x-y| \geq 0$

$\Rightarrow d(x,y) \geq 0$.

$d(x,y) = 0 \Leftrightarrow |x-y| = 0$

$\Leftrightarrow x-y = 0$

$\Leftrightarrow x = y$.

ii) $d(x,y) = |x-y| = |y-x| = d(y,x)$

$\therefore d(x,y) = d(y,x)$.

iii) $d(x,z) = |x-z| = |x-y+y-z|$

$\leq |x-y| + |y-z|$

$= d(x,y) + d(y,z)$.

$\therefore d(x,z) \leq d(x,y) + d(y,z)$.

Hence $d$ is metric.

Claim,

Standard metric $d$ induces ordered topology.

Let $\mathcal{B} = \{B_d(x,\varepsilon) / x \in X, \varepsilon > 0\}$ be basis for the standard metric topology. and $\mathcal{B}'$ be the basis for the ordered topology.

Let $\mathcal{T}$ and $\mathcal{T}'$ be the topology for $\mathcal{B}$ and $\mathcal{B}'$ respectively,

Claim $\mathcal{T} = \mathcal{T}'$

i.e, claim $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T}' \subset \mathcal{T}$.

let $(a,b) \in \mathcal{B}'$

take $x = \frac{a+b}{2}$ and $\varepsilon = \frac{b-a}{2}$.

Let $y \in (a,b)$ then $a < y < b$.

Now, $d(x,y) = |x-y|$

$= \left| \frac{a+b}{2} - y \right|$

$< \left|\frac{a+b}{2} - a\right| = \frac{b-a}{2}$

$\therefore d(x,y) < \varepsilon \Rightarrow y \in B_d(x,\varepsilon)$.

$\therefore (a,b) \subset B_d(x,\varepsilon)$

ie, $B' \subset B$.

$\mathcal{J} \subset \mathcal{J}' \rightarrow (1)$ [By Lemma 1.2.2]

Consider the ball $B_d(x,\varepsilon) = \{y / d(x,y) < \varepsilon\}$

$= \{y / |x-y| < \varepsilon\}$

$= \{y / -\varepsilon < x-y < \varepsilon\} = \{y / -x-\varepsilon < -y < -x+\varepsilon\}$

$= \{y / x+\varepsilon > y > x-\varepsilon\}$

$= \{y / x-\varepsilon < y < x+\varepsilon\}$

$\therefore B \subset B'$

$\mathcal{J}' \subset \mathcal{J} \rightarrow (2)$ [by Lemma 1.2.2]

From (1) & (2) we get

$\mathcal{J} = \mathcal{J}'$

---

Standard bounded Metric.

Let $X$ be a metric space with metric $d$. Then $\bar{d}: X \times X \rightarrow \mathbb{R}$ defined by $\bar{d}(x,y) = \min\{d(x,y), 1\}$, $\bar{d}$ is the standard bounded metric of $d$.

pf:-

Claim 1,

$\bar{d}$ is Metric.

i, $\bar{d}(x,y) = \min\{d(x,y), 1\}$

$= d(x,y)$ or $1$

$\geq 0$. ($\because d$ is metric)

$\bar{d}(x,y) \geq 0$.

ii, $\bar{d}(x,y) = 0 \Leftrightarrow \min\{d(x,y), 1\} = 0$.

$\Leftrightarrow d(x,y) = 0$.

$\Leftrightarrow x = y$. ($\because d$ is Metric)

iii, $\bar{d}(x,y) = \min\{d(x,y), 1\}$

$= \min\{d(y,x), 1\}$

$\bar{d}(x,y) = \bar{d}(y,x)$

Claim $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

Case i,

Either $d(x,y) \geq 1$ (or) $d(y,z) \geq 1$

Let $d(x,y) \geq 1$.

then $\bar{d}(x,y) = \min\{d(x,y), 1\}$

$$\therefore \bar{d}(x,y) + \bar{d}(y,z) = 1 + \bar{d}(y,z) \quad [\bar{d}(x,y) = 1]$$
$$\geq 1 \rightarrow (1)$$

But $\bar{d}(x,z) = \min\{d(x,z), 1\} \leq 1 \rightarrow (2)$

From (1) & (2)

$$\therefore \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z).$$

Similarly, if $\bar{d}(y,z) > 1$.

$$\Rightarrow \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z).$$

~~Claim~~ case ii,

Let $d(x,y) < 1$ and $d(y,z) < 1$.

$d(x,y) < 1 \Rightarrow \bar{d}(x,y) = \min\{d(x,y), 1\}$
$= d(x,y) \rightarrow (3)$

$d(y,z) < 1 \Rightarrow \bar{d}(y,z) = \min\{d(y,z), 1\}$
$= d(y,z) \rightarrow (4)$

Now, $\bar{d}(x,y) + \bar{d}(y,z) = d(x,y) + d(y,z)$
$\geq d(x,z) \rightarrow (5)$.

By defn. of $\bar{d}$

$$\bar{d}(x,z) \leq d(x,z) \rightarrow (6)$$

From (5) & (6)

$$\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z).$$

Thus in both cases, $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

Claim II,

$d$ and $\bar{d}$ induces the same topology of $d$.

Let $\mathcal{T}$ be the topology induced by $d$ and $\mathcal{T}'$ be the topology induced by $\bar{d}$.

Claim, $\mathcal{T} = \mathcal{T}'$

Let $\mathcal{B} = \{B_d(x,\varepsilon) / x \in X, \varepsilon > 0\}$ be a basis for the topology on $X$ and

$$\mathcal{B}' = \{B_{\bar{d}}(x,\varepsilon) / x \in X, \varepsilon > 0\}$$

Claim $\mathcal{T}' \subset \mathcal{T}$

Let $U \in \mathcal{T}'$ and let $x \in U$.

$U \in \mathcal{T}'$ is generated by $\mathcal{B}'$

$\Rightarrow$ there exist $B_{\bar{d}}(x,\varepsilon) \in \mathcal{B}'$ such that

$x \in B_{\bar{d}}(x,\varepsilon) \subset U \rightarrow (1)$.

Now, prove that $B_d(x,\varepsilon) \subset B_{\bar{d}}(x,\varepsilon)$

Let $z \in B_d(x,\varepsilon) \Rightarrow d(x,z) < \varepsilon$.

But $\bar{d}(x,z) \leq d(x,z)$.

$\bar{d}(x,z) < \varepsilon \Rightarrow z \in B_{\bar{d}}(x,\varepsilon)$

$B_d(x,\varepsilon) \subset B_{\bar{d}}(x,\varepsilon) \subset U$ [By (1)]

$\therefore x \in B_d(x,\varepsilon) \subset U$

$\Rightarrow U \in \mathcal{J}$

$\mathcal{J}' \subset \mathcal{J} \rightarrow (2)$

Now claim $\mathcal{J} \subset \mathcal{J}'$

Let $V \in \mathcal{J}$ and $x \in V$.

Since $\mathcal{J}$ is generated by $\mathcal{B}$

$\Rightarrow$ there exist a $B_d(x,\varepsilon) \subset \mathcal{B}$.

Such that $x \in B_d(x,\varepsilon) \subset V \rightarrow (3)$

Let $\delta = \min\{1,\varepsilon\}$.

Now, prove $B_{\bar{d}}(x,\delta) \subset B_d(x,\varepsilon)$

Let $z \in B_{\bar{d}}(x,\delta)$.

$\Rightarrow \bar{d}(x,z) < \delta$.

$\Rightarrow d(x,z) < \delta$.

$\Rightarrow d(x,z) < \varepsilon$ $\{\because \delta = \min\{1,\varepsilon\}\}$.

$\Rightarrow z \in B_d(x,\varepsilon)$.

$B_{\bar{d}}(x,\delta) \subset B_d(x,\varepsilon) \subset V$ [By (3)]

$x \in B_{\bar{d}}(x,\delta) \subset V$

$V \in \mathcal{J}'$

$\mathcal{J} \subset \mathcal{J}' \rightarrow (4)$.

From (2) & (4)

$\mathcal{J} = \mathcal{J}'$

---

Lemma 9.2

(i) Let $d$ and $d'$ be the matrices on the set $X$. Let $\mathcal{J}$ and $\mathcal{J}'$ be two topological induces respectively $d$ and $d'$ then $\mathcal{J}'$ is finer then $\mathcal{J}$. iff for $x \in X$ and for each $\varepsilon > 0$ there exist $\delta > 0$ such that $B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$.

Pf:- (22) Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologies $\mathcal{J}$ and $\mathcal{J}'$ respectively on $X$. Then the following are equivalent. i) $\mathcal{J}'$ is finer than $\mathcal{J}$. ii) for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing $x$, there is a basis element $B' \in \mathcal{B}'$ $\ni$: $x \in B' \subset B$.

Let $\mathcal{J}'$ be finer than $\mathcal{J}$.

Also, Let $B_d(x, \epsilon)$ be a basis element for $\mathcal{J}$.

By Knows Lemma, 13.3.

there exist a basis element $B'$ for the topology $\mathcal{J}'$ such that $x \in B' \subset B_d(x, \epsilon)$.

We can find a ball,

$B_{d'}(x, \delta)$ such that $x = B_{d'}(x, \delta) \subset B' \subset B_d(x, \epsilon)$.

for each $x \in X$, for each $\epsilon > 0$.

there exist $\delta > 0$ such that,

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon).$$

Conversely,

Assume that $\delta$-$\epsilon$ condition,

ie, for every $\epsilon > 0$ $\forall x \in X$. there exist $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

claim,

$\mathcal{J}' \supset \mathcal{J}$.

ie, to claim, given a basis element $B$ for $\mathcal{J}$ containing $x$, We can find a basis element $B'$ for $\mathcal{J}'$ such that $B' \subset B$.

Let $B$ be a basis element for the topology $\mathcal{J}$,

then We can find a ball $B_d(x, \epsilon)$ which is in $B$ containing $x$.

$\therefore x \in B_d(x, \epsilon) \subset B$.

By given for every $\epsilon > 0$ there exist $\delta > 0$ such that,

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon).$$

$\therefore$ there exist $B' = B_{d'}(x, \delta)$ for $\mathcal{J}'$ such that $B' \subset B$.

$\therefore \mathcal{J}' \supset \mathcal{J}$.

---

Defn:-

Euclidean Metric in $\mathbb{R}^n$.

Let $x = (x_1, x_2 \ldots x_n) \in \mathbb{R}^n$.

W.K.T,

$$\|x\| = (x_1^2+x_2^2+\dots+x_n^2)^{1/2}$$
$$= \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

We define the Euclidean d by

$$d(x,y) = \|x-y\|$$
$$= \left[\sum_{i=1}^{n} (x_i-y_i)^2\right]^{1/2} \quad \forall\ x_i, y_i \in \mathbb{R}.$$

Pf: (d is metric)

$$d(x,y) = \|x-y\| = \left(\sum_{i=1}^{n} (x_i-y_i)^2\right)^{1/2} \geq 0 \quad \forall\ x_i, y_i \in \mathbb{R}.$$

$$d(x,y) = 0 \iff \left[\sum_{i=1}^{n} (x_i-y_i)^2\right]^{1/2} = 0.$$

$$\iff (x_i-y_i)^2 = 0 \quad \forall\ i = 1,2\dots n.$$
$$\iff x_i = y_i$$
$$\iff (x_1, x_2 \dots x_n) = (y_1, y_2 \dots y_n)$$
$$\iff x = y.$$

//

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i-y_i)^2\right)^{1/2} = \left(\sum_{i=1}^{n} (y_i-x_i)^2\right)^{1/2}$$
$$= d(y,x).$$

$$\|x-z\| = \|x-y+y-z\|, \quad x,y \in \mathbb{R}^n.$$
$$\|x-z\| \leq \|x-y\| + \|y-z\|.$$
$$\left(\sum_{i=1}^{n} (x_i-z_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} (x_i-y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n} (y_i-z_i)^2\right)^{1/2}.$$

$$d(x,z) \leq d(x,y) + d(y,z)$$

$\therefore$ d is a metric.

---

Square Metric.

Defn:

Let $x, y \in \mathbb{R}^n$ then the Square Metric $\rho$ is defined as,

$$\rho(x,y) = \max\{|x_i-y_i|\}, \quad i = 1,2\dots n.$$

Pf:- ($\rho$ is metric)

Given $\rho(x,y) = \max\{|x_i-y_i|\}, \quad i = 1,2\dots n.$

$$\rho(x,y) = \max\{|x_1-y_1|, |x_2-y_2| \dots |x_n-y_n|\} \geq 0.$$

$$\rho(x,y) = 0 \iff \max\{|x_1-y_1|, |x_2-y_2| \dots |x_n-y_n|\} = 0.$$

$$\iff x_i - y_i = 0 \quad \forall\ i = 1,2\dots n.$$
$$\iff x_i = y_i \quad \forall\ i = 1,2\dots n.$$
$$\iff (x_1, x_2 \dots x_n) = (y_1, y_2 \dots y_n)$$
$$\iff x = y.$$

$\rho(x,y) = \text{Max}\{|x_i - y_i|\} = \max\{|y_i - x_i|\}$
$= \rho(y,x)$.

$|x_i - z_i| = |x_i - y_i + y_i - z_i| \quad \forall\ x_i, y_i, z_i \in \mathbb{R}$.

$|x_i - z_i| \le |x_i - y_i| + |y_i - z_i|$

$\max\{|x_i - z_i|\} \le \text{Max}\{|x_i - y_i| + |y_i - z_i|\}$

$\le \text{Max}\,|x_i - y_i| + \max\,|y_i - z_i|$

$\rho(x,z) \le \rho(x,y) + \rho(y,z)$

$\therefore \rho$ is a metric on $\mathbb{R}^n$.

Properties of Norm :-

* $\|x\| \geq 0$.
* $\|x\| = 0$ iff $x = 0$
* $\|x - y\| \le \|x\| + \|y\|$ (Minkowski inequality)
* Cauchy's inequality,

Let $x = (x_1, \ldots x_n) \in \mathbb{R}^n$ and
$y = (y_1 \ldots y_n) \in \mathbb{R}^n$

Then $\sum_{i=1}^{n} |x_i - y_i| \le \|x\|\,\|y\|$.

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10m Thm : 9.3 (*)

The topological space in $\mathbb{R}^n$ induced by the Euclidean metric $d$ and the square metric $\rho$ are the same as the product topology on $\mathbb{R}^n$.

Pf :-

Define proving thm, Let us prove the following Lemma.

"Let $x, y \in \mathbb{R}^n$ define $\rho(x,y) \le d(x,y) \le \sqrt{n}\,\rho(x,y)$."

Pf :-

$$|x_i - y_i|^2 = (x_i - y_i)^2$$
$$\le \sum_{i=1}^{n} (x_i - y_i)^2$$
$$= (d(x,y))^2 \quad \forall\ i = 1, 2 \ldots n$$
$$|x_i - y_i|^2 \le (d(x,y))^2 \quad \forall\ i = 1, 2 \ldots n.$$
$$|x_i - y_i| \le d(x,y)$$
$$\text{Max}\,|x_i - y_i| \le d(x,y) \rightarrow (1)$$

Also, $|x_i - y_i| \le \text{Max}\{|x_i - y_i|\}, \; i = 1, 2 \ldots n$.

$|x_i - y_i| \le \rho(x,y)$

$|x_i - y_i|^2 \leq (\rho(x,y))^2$

$\sum_{i=1}^{n} |x_i - y_i|^2 \leq \sum_{i=1}^{n} (\rho(x,y))^2$

$(d(x,y))^2 \leq n(\rho(x,y))^2$

$\therefore d(x,y) \leq \sqrt{n}\, \rho(x,y) \longrightarrow (2)$

from (1) and (2)

$\rho(x,y) \leq d(x,y) \leq \sqrt{n}\, \rho(x,y)$

claim:-

Topology induced by the Euclidean Metric $d$ is same as the topology induced by square Metric $\rho$.

It is enough to prove that, $J = J'$

Where $J$ is the topology induced by the Metric $d$.

$J'$ is the topology induced by $\rho$.

i.e, claim,

$J'$ is finer than $J$ and $J$ is finer then $J'$

first claim, $J' \subset J$.

Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$.

claim,

$B_d(x,\varepsilon) \subset B_\rho(x,\varepsilon)$.

Let $z \in B_d(x,\varepsilon)$

$\Rightarrow d(x,z) < \varepsilon \longrightarrow (3)$

By the result, $\rho(x,z) \leq d(x,z) < \varepsilon \longrightarrow (4)$

from (3) and (4)

$\rho(x,z) < \varepsilon$.

$\Rightarrow z \in B_\rho(x,\varepsilon)$

$\therefore B_d(x,\varepsilon) \subset B_\rho(x,\varepsilon)$

$\Rightarrow J \supset J'$

Next we claim, $J \subset J'$

$B_\rho\left(x, \frac{\varepsilon}{\sqrt{n}}\right) < B_d(x,\varepsilon)$

Let $z \in B_\rho\left(x, \frac{\varepsilon}{\sqrt{n}}\right)$

$\Rightarrow \rho(x,z) < \varepsilon/\sqrt{n} \Rightarrow \sqrt{n}\,\rho(x,z) < \varepsilon \longrightarrow (5)$

By the result,

WKT, $d(x,z) \leq \sqrt{n}\,\rho(x,z) \longrightarrow (6)$

from (5) & (6), We have $d(x,z) < \varepsilon$. (26)

$\Rightarrow z \in B_d(x,\varepsilon)$

$\Rightarrow B_\rho\left(x, \frac{\varepsilon}{\sqrt{n}}\right) \subset B_d(x,\varepsilon)$

$\Rightarrow \mathcal{J}' \supset \mathcal{J}$.

ie, The topology induced by $d$ equal to the topology induced by $\rho$. $\mathcal{J} = \mathcal{J}'$

claim 3

Let us prove the topology induced by $\rho$ is same as the product topology on $\mathbb{R}^n$.

Let $\mathcal{B} = \{(a_1,b_1) \times (a_2,b_2) \times \dots (a_n,b_n) / (a_i,b_i) \in \mathbb{R}\}$ be the basis for the product topology on $\mathbb{R}^n$.

Let $B \in \mathcal{B}$ such that $B = (a_1,b_1) \times (a_2,b_2) \times \dots (a_n,b_n)$

Let $x \in B$.

$\Rightarrow (x_i)_{i=1}^n \subset (a_1,b_1) \times (a_2,b_2) \times \dots (a_n,b_n)$.

$\Rightarrow x_i \in (a_i,b_i) \; \forall \; i=1,\dots n$.

Now, for each $i$, there is an $\varepsilon > 0$ such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i)$ 14

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2 \dots \varepsilon_n\}$

then $\rho \le \varepsilon_i \; \forall \; i$

$\Rightarrow (x_i - \varepsilon, x_i + \varepsilon) \subset (x_i - \varepsilon_i, x_i + \varepsilon_i) \; \forall \; i = 1, \dots n$

$\subset (a_i, b_i) \; \forall \; i = 1, \dots n$.

$\Rightarrow (x_i - \varepsilon, x_i + \varepsilon) \subset \prod_{i=1}^{n} (x_i - \varepsilon_i, x_i + \varepsilon_i)$

$\subset \prod_{i=1}^{n} (a_i, b_i)$

$\Rightarrow B_\rho(x,\varepsilon) \subset B$

$\Rightarrow$ The topology induced by $\rho$ is finer than the product topology.

Conversely Let $B_\rho(x,\varepsilon)$ be a basis element for $\rho$-topology.

Let $y \in B_\rho(x,\varepsilon)$.

We need to find a basis element $B$ of the product topology such that,

$x \in B \subset B_\rho(x,\varepsilon)$

Now, $B_\rho(x,\varepsilon) = \{y / \rho(x,y) < \varepsilon\}$

$= \{y / \max |x_i - y_i| < \varepsilon \; \forall \; i = 1, 2 \dots n\}$

$= \{y_i / |x_i - y_i| < \varepsilon \; \forall \; i\}$

$= \{y_i / y_i \in (x_i - \varepsilon, x_i + \varepsilon) \; \forall \; i\}$

$= \{y / y \in \prod (x_i - \varepsilon, x_i + \varepsilon)\}$

$= \prod (x_i - \varepsilon, x_i + \varepsilon)$

which is basis element for the product topology.
The product topology is finer than $\rho$-topology.

But already we have proved topology induced by $d$ equal to the topology induced by $\rho$.

Hence, the metrics $d$ and $\rho$ both induces the Product topology on $\mathbb{R}^n$

Hence proved.

**Note:-**

By the above thm, we know that $\mathbb{R}^n$ is Metrizable space.

The thm is called Metrization theorem. The metrics on $\mathbb{R}^n$ generalized the Euclidean metric and square metric in the space $\mathbb{R}^\omega$, we get

$$d(x,y) = \left[ \sum_{i=1}^{\omega} (x_i - y_i)^2 \right]^{1/2}.$$

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$$\rho(x,y) = \text{lub}\{d(x_i,y_i)\}, \quad i=1,2\ldots$$

By these formular, do not make sense on $\mathbb{R}^\omega$, the series need not converges and the sets may not be bounded to avoid the difficult in the metric change the metric $d(x,y) = |x-y|$ by $\bar{d}$.

ie, the standard bounded metric

$$\bar{d}(x,y) = \min\{d(x,y), 1\}$$
$$= \min\{|x-y|, 1\}$$

For $x,y \in \mathbb{R}^\omega$. The metric $\bar{\rho}$ is defined as $\bar{\rho}(x,y) = \text{lub}\{\bar{d}(x_i,y_i)\}$.

## Uniform Metric on $\mathbb{R}^J$:

**Defn:-**

Given an index set $J$ and given point $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$. The metric $\bar{\rho}$ on $\mathbb{R}^J$ is defined as $\bar{d}$. Where $\bar{d}$ is the standard bounded metric of $d$ on $\mathbb{R}$. The metric $\bar{\rho}$ is called the Uniform metric on $\mathbb{R}^J$ and the topology it induces is called the Uniform topology.

*. Thm 9.4



The uniform topology in $R^J$ is finer than the product topology on $R^J$. Where $J$ is countable.

[Note: They are different if $J$ is infinite].

Pf:-

The Uniform Metric in $R^J$ is defined as

$\bar{\rho}(x,y) = \text{lub}\{\bar{d}(x_\alpha, y_\alpha)\}_{\alpha \in J}$.

Let $\tau$ and $\tau'$ be the product topology and Uniform topology on $R^J$ respectively.

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for $\tau$ and $\tau'$ respectively.

Claim $\tau' \supset \tau$.

For that claim, $\forall\ B \in \mathcal{B}$ there exist an element $B' \in \mathcal{B}'$ such that $B' \subset B$.

Let $U = \pi U_\alpha$ be any basis element of the product topology.

By defn. of product topology,

$U = \pi U_\alpha$.

$\Rightarrow U_\alpha$ is open in $R$ and $U_\alpha = R$ except for a finite number of places.

Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be the n indices for which $U_\alpha \neq R$.

Let $x = (x_\alpha)_{\alpha \in J} \in U$.

Now, for each $\alpha_i$, $i = 1,2 \ldots n$ choose $\varepsilon_i$ such that

$(x_{\alpha_i} - \varepsilon_i, x_{\alpha_i} + \varepsilon_i) \subset U_{\alpha_i} \longrightarrow (1)$

Without loss of generality.

let $\varepsilon_i < 1$, $\forall i$.

claim, $B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$ $\forall i = 1,2 \ldots n$

let $a \in B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i)$

$\Rightarrow \bar{d}(x_{\alpha_i}, a) < \varepsilon_i$

$\Rightarrow \min\{d(x_{\alpha_i}, a)\} < \varepsilon_i$

$\Rightarrow d(x_{\alpha_i}, a) < \varepsilon_i$ $[\because \varepsilon_i < 1]$

$a \in (x_{\alpha_i} - \varepsilon_i, x_{\alpha_i} + \varepsilon_i) \subset U_{\alpha_i}$ $\forall i = 1,2 \ldots n$

$a \in U_{\alpha_i}$, $\forall i = 1,2 \ldots n$ by (1)

$B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$, $\forall i = 1,2 \ldots n$

$\hookrightarrow (2)$

claim/

Choose $B_{\bar{\rho}}(x, \varepsilon) \subset U$.

Let $z \in B_{\bar{\rho}}(x, \varepsilon)$

$\Rightarrow \bar{\rho}(x, z) < \varepsilon$

$\Rightarrow lub\{\bar{d}(x_\alpha, z_\alpha)\}_{\alpha \in J} < \varepsilon$

$\Rightarrow \bar{d}(x_\alpha, z_\alpha) < \varepsilon \;\forall\; \alpha.$

$\Rightarrow z_\alpha \in B_{\bar{d}}(x_\alpha, \varepsilon) \subset B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$ [by (*)]

$\Rightarrow z_\alpha \in U_{\alpha_i} \;\forall\; \alpha = \alpha_1 \ldots \alpha_n$

ie, $z_\alpha \in U_{\alpha_i}$ for $\alpha = \alpha_1, \alpha_2 \ldots \alpha_n$

and $z_\alpha \in R$ for other values of $\alpha$.

$z \in \pi U_\alpha$

$\Rightarrow z \in U$

$B_{\bar{\rho}}(x, \varepsilon) \in U$

$\Rightarrow$ There exist a basis $B_{\bar{\rho}}(x, \varepsilon)$ contains in a basis elements $U$ of the product topology in $R^J$.

$\Rightarrow$ Uniform topology is finer than the product topology.

Theorem : 9.5

$R^\omega$ is metrizable under the metric $D$. Let $\bar{d}(a,b) = \min\{d(a,b), 1\}$ be the standard bounded metric on $R$. If $x, y$ are any two points of $R^\omega$, the metric $D$ on $R^\omega$ is defined as $D(x,y) = lub\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$.

Then $D$ is metric that induces the product topology.

Pf:-

Claim $D$ is metric.

i, first to prove, $D(x,y) \geq 0$.

$\bar{d}$ is a metric on $R$.

$\Rightarrow \bar{d}(x_i, y_i) \geq 0 \;\forall\; i$

$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} \geq 0 \;\forall\; i$

$\Rightarrow lub\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} \geq 0.$

$\Rightarrow D(x,y) \geq 0.$

ii, Let $D(x,y) = 0$.

$lub\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} = 0 \Leftrightarrow \bar{d}(x_i, y_i) = 0 \;\forall\; i.$

$\Leftrightarrow x_i = y_i \;\forall\; i$, $\bar{d}$ is metric $\Leftrightarrow x = y$.

iii, $D(x,y) = l.u.b\left\{\frac{\bar{d}(x,y)}{i}\right\}$ ($\because \bar{d}$ is metric)

$= D(y,x)$

Thus $D(x,y) = D(y,x)$

Claim ii, $D(x,z) \le D(x,y) + D(y,z)$.

We know, $\bar{d}(x_i, z_i) \le \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i)$

$$\Rightarrow \frac{\bar{d}(x_i, z_i)}{i} \le \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i}$$

$$\le D(x,y) + D(y,z)$$

[$\because$ each $i$ every element of the collection

$$\frac{\bar{d}(x_i, y_i)}{i} \le \sup \frac{\bar{d}(x_i, y_i)}{i}$$

$$\Rightarrow \text{lub}\left(\frac{\bar{d}(x_i, y_i)}{i}\right)$$

$$\le D(x,y) + D(y,z).$$

$$\Rightarrow D(x,z) \le D(x,y) + D(y,z)$$

$\therefore D$ is a metric.

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Next claim, $R^w$ is metrizable.

ie, claim the topology induces by $D$ coincides with the original topology on $R^w$ namely the product topology. In short the claim is $D$ induces the product topology.

Proof of the claim, let $\mathcal{T}$ be the original topology namely the product in $R^w$.

Let $\mathcal{T}'$ be the topology induced by the metric in $R^w$. Our claim $\mathcal{T} = \mathcal{T}'$

For that claim,

$\mathcal{T}$ is finer than $\mathcal{T}'$ & $\mathcal{T}'$ is finer than $\mathcal{T}$.

First claim $\mathcal{T} \subset \mathcal{T}'$.

Let $U$ be an open set in the metric topology and $x \in U$.

We have to find an open set $V$ in the product topology. such that $x \in V \subset U$.

Choose an $\varepsilon$-ball $B_D(x, \varepsilon) \to (1)$ lying in $U$.

Then choose $N$ large enough such that

$\frac{1}{N} < \varepsilon$.

Let $V$ be the basis for the product topology.
and $V = (x_1-\varepsilon, x_1+\varepsilon) \times (x_2-\varepsilon, x_2+\varepsilon) \times \dots (x_N-\varepsilon, x_N+\varepsilon) \times R \times R \times \dots$
We claim that, $V \subset B_D(x,\varepsilon)$. (3)

Let $y = (y_1, y_2 \dots y_N) \in V$
Now, by defn,

$$\bar{d}(x_i, y_i) = \min\{d(x_i,y_i), 1\}$$
$$\Rightarrow \bar{d}(x_i,y_i) \le 1$$
$$\Rightarrow \frac{\bar{d}(x_i,y_i)}{i} \le \frac{1}{i}$$

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$\therefore$ for $i \ge N$,

$$\frac{\bar{d}(x_i,y_i)}{i} < \frac{1}{N}$$

Defn. of $D$, $D(x,y) = \text{lub}\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$

$$\Rightarrow D(x,y) \le \text{Max}\left\{\frac{\bar{d}(x_1,y_1)}{1}, \frac{\bar{d}(x_2,y_2)}{2}, \dots \frac{\bar{d}(x_N,y_N)}{N}, \frac{1}{N}\right\} \to (2)$$

$\left\{\because \frac{\bar{d}(x_i,y_i)}{i} \le \frac{1}{N} \text{ for } i \ge N\right\}$

Our claim is $V \subset B_d(x,\varepsilon)$ so that $y \in V$.

$\Rightarrow (y_1, y_2 \dots) \in (x_1-\varepsilon, x_1+\varepsilon)(x_2-\varepsilon, x_2+\varepsilon) \dots (x_N-\varepsilon, x_N+\varepsilon) \times R \times R$.

$\Rightarrow y_i \in (x_i-\varepsilon, x_i+\varepsilon)$ for $i = 1$ to $N$ & $y_i \in R$ for $i = N$.

$\Rightarrow d(x_i,y_i) < \varepsilon$ for $i = 1$ to $N$.

WKT,

$$\bar{d}(x_i,y_i) \le d(x_i,y_i) \text{ always}$$

$\therefore \bar{d}(x_i,y_i) < \varepsilon$ for $i = 1$ to $N$.

$\Rightarrow \frac{\bar{d}(x_i,y_i)}{i} < \varepsilon$ for $i \le N$ $(i = 0)$ $\left(\because \bar{d} < \varepsilon \Rightarrow \frac{\bar{d}}{i} < \varepsilon\right)$

(2) becomes,

$$D(x,y) \le \text{Max}\left\{\frac{\bar{d}(x_1,y_1)}{1}, \frac{\bar{d}(x_2,y_2)}{2} \dots \frac{\bar{d}(x_N,y_N)}{N}, \frac{1}{N}\right\}$$
$$\le \text{Max}(\varepsilon, \varepsilon \dots \varepsilon) \quad (\because 1/N < \varepsilon)$$

$\therefore D(x,y) < \varepsilon$.

$\Rightarrow y \in B_D(x,\varepsilon)$

$\therefore V \subset B_D(x,\varepsilon)$. $\quad [\because y \in V \Rightarrow y \in B_D(x,\varepsilon)$, $V \subset B_D(x,\varepsilon)]$

By (1), $B_D(x,\varepsilon) \subset U$.

$\therefore V \subset U \Rightarrow \mathcal{J} \subset \mathcal{J}' \to$ (I)

Next claim Conversely $\mathcal{J}' \subset \mathcal{J}$.

Consider basis element $U = \prod_{i \in I} U_i$ for the product topology where $U_i$ is open in $\mathbb{R}$ for $i = \alpha_1 \ldots \alpha_n$, $U_i = \mathbb{R}$ for the other indices. (32)

Let $x \in U$.

We find an open set $V$ of the metric topology $\ni$: $x \in V \subset U$, whose an interval $(x_i - \varepsilon, x_i + \varepsilon)$ in $\mathbb{R}$ centered about $x_i$ $\ni$ lying in $U_i$ for $i = \alpha_1 \ldots \alpha_n$.

ie, choose $\varepsilon_1 \ldots \varepsilon_n$ be $n$ positive real number. $\ni$: $(x_1 - \varepsilon_1, x_1 + \varepsilon_1) \subset U_{\alpha_1}, \ldots$

$(x_2 - \varepsilon_2, x_2 + \varepsilon_2) \subset U_{\alpha_2}$

$(x_n - \varepsilon_n, x_n + \varepsilon_n) \subset U_{\alpha_n}$.

Also choose each $\varepsilon_i < 1$.

Then define $\varepsilon_1 = \min \left\{ \frac{\varepsilon_i}{i} / i = 1 \text{ to } n \right\}$

We claim that, $B_D(x, \varepsilon_1) \subset U$.

Let $y \in B_D(x, \varepsilon_1) \Rightarrow D(x, y) < \varepsilon$

$\Rightarrow \text{lub} \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\} < \varepsilon$

$\Rightarrow \bar{d}(x_i, y_i) < \varepsilon_i / i$ $\left\{ \because \varepsilon = \min \varepsilon_i / i \,/\, i = 1 \text{ to } n \Rightarrow \varepsilon_1 = \frac{\varepsilon_i}{i} \right\}$

$\Rightarrow \min \{ d(x_i, y_i), 1 \} < \varepsilon_i$

$\Rightarrow \min \{ |x_i - y_i|, 1 \} < \varepsilon_i$

$\Rightarrow |x_i - y_i| < \varepsilon_i$ $\quad (\because \varepsilon_i < 1)$

$\Rightarrow y_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i)$ $\forall i = \alpha_1, \alpha_2 \ldots \alpha_n$.

$\Rightarrow y_i \in U_i$ $\forall i = \alpha_1 \ldots \alpha_n$.

$\Rightarrow y \in \prod_{i \in \gamma} U_i$, where $U_i$ is open in $\mathbb{R}$ $\forall i$ & $U_i = \mathbb{R}$ except for $i = \alpha_1 \ldots \alpha_n$.

$\therefore B_D(x, \varepsilon_1) \subset \prod_{i \in \gamma} U_i = U$.

$\Rightarrow \mathcal{J}' \subset \mathcal{J} \rightarrow \text{II}$

From (I) and (II) we have

$\mathcal{J} = \mathcal{J}'$

$\therefore$ The topology induced by $D$ coincide with the original topology.

$\mathbb{R}^{\omega}$ is metrizable.

Thm 10.1

Let $f: X \to Y$. Let $X$ and $Y$ be Metrizable With Metrics $d_X$ and $d_Y$ respectively. Then the Continuity of $f$ is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Pf:

Let $X$ and $Y$ be Metrizable Spaces and Let $f: X \to Y$ be Continuous.

Claim,

for given $\varepsilon > 0$, there exist $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

~~It is enough to prove $f(\overline{A}) \subset \overline{f(A)}$~~

Now Consider $x \in X$ and a positive number $\varepsilon > 0$.

then, $B(f(x), \varepsilon)$ is open in $Y$.

Since $f: X \to Y$ is continuous.

$f^{-1}(B(f(x), \varepsilon))$ is open in $X$ and Containing $x$.

there exist a $\delta$-ball $B(x, \delta)$ such that,

$B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$

$\Rightarrow f(B(x, \delta)) \subset B(f(x), \varepsilon) \to (1)$

Let $y \in B(x, \delta) \Rightarrow d_X(x, y) < \delta$.

Then $f(y) \in f(B(x, \delta)) \subset B(f(x), \varepsilon)$ [by (1)]

$\Rightarrow f(y) \in B(f(x), \varepsilon)$

$\Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Conversely,

Assume $\varepsilon$-$\delta$ Condition.

Claim: $f: X \to Y$ is Continuous.

Let $V$ be an open set in $Y$.

ie, to claim $f^{-1}(V)$ is open in $X$.

Let $x \in f^{-1}(V) \Rightarrow f(x) \in V$.

Since $V$ is open in $Y$ there exist a ball $B(f(x), \varepsilon)$ is open in $Y$ for some $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset V$.

By $\varepsilon$-$\delta$ Condition there exist a $\delta$-ball Such that $f(B(x, \delta)) \subset B(f(x), \varepsilon) \subset V$

$\Rightarrow f(B(x,\delta)) \subset V$ (34)

$\Rightarrow B(x,\delta) \subset f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ is open in X.

$\Rightarrow f$ is Continuous.

Convergence of sequence :-

A sequence $(x_1, x_2 \dots x_n \dots)$ of Points of X is said to converges to the point $x$ of X for every neighbourhood U of $x$ there exist a positive integer N such that $x_i \in U \ \forall\ i \geq N$.

Ex. Thm 10.2

## Sequence Lemma

Let X be a topology space and let A subset of X. If there is a sequence of points of A converging to $x$. then $x \in \bar{A}$. The converse holds. If X is Metrizable.

Pf:

Let $(x_n)$ be a sequence of point of A. Converging to $x$.
Claim, $x \in \bar{A}$

Now, $(x_n) \to x$ where $x_n \in A$.

$\Rightarrow$ for every U of $x$ there exist a positive integer N such that $x_i \in U \ \forall\ i \geq N$. (by defn of Converges)

$\Rightarrow$ U contains some points of A.

$\Rightarrow$ Every neighbourhood U of $x$ intersect A.

$\Rightarrow x \in \bar{A}$

Conversely,

Let X be a Metrizable and let $x \in \bar{A}$

Claim, there exist a sequence of Points of A converging to $x$.

Since, X is Metrizable there exist a Metric d in X which induces the Orginal topology of X.

Now for each Positive integer n, choose a neighbourhood $B_d(x, 1/n)$ of $x$ with radius $1/n$.

Now $x \in \bar{A}$

$\Rightarrow$ Every neighbourhood of $x$ intersects A.

$\Rightarrow B_d(x, 1/n) \cap A \neq \phi$.

$\therefore$ For each integer n, Choose a point $x_n$ such that $x_n \in B_d(x, 1/n) \cap A$.

Thus, we get a sequence $\{x_n\}$ of points of A.

Next claim that, the sequence $\{x_n\}$ of points of A ~~Containing~~ Converging to $x$. (35)

For that, let U be a neighbourhood of $x$.

Now, U is open and $x \in U$.

$\Rightarrow$ for given $\varepsilon > 0$ there exist a ball $B_d(x, \varepsilon)$ such that $x \in B_d(x, \varepsilon) \subset U \rightarrow (1)$

Choose a positive integer N such that $\frac{1}{N} < \varepsilon$.

For $i \geq N$, $\frac{1}{i} \leq \frac{1}{N}$

$\therefore B_d(x, \frac{1}{i}) \subset B_d(x, \frac{1}{N}) \quad \forall i \geq N$.

Now, $x_i \in B_d(x, \frac{1}{i}) \subset B_d(x, 1/N)$

$\Rightarrow x_i \in B_d(x, 1/N) \in B_d(x, \varepsilon) \quad \because (1/N < \varepsilon)$

$\Rightarrow x_i \in B_d(x, \varepsilon) \subset U$

$\Rightarrow x_i \in U \quad \forall i \geq N$.

$\Rightarrow \{x_n\} \rightarrow x$.

---

*. Thm: 10.3

Let $f: X \rightarrow Y$ and (Let X and Y be a Metrizable space) Then function $f$ is continuous $\Leftrightarrow$ for every convergence sequence $\{x_n\}$ converges to $x$ in X, the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y.

Pf:

Let $f: X \rightarrow Y$ be continuous and let X be a Metrizable. Also assume that,

$\{x_n\} \rightarrow x$ in X.

Claim, $\{f(x_n)\} \rightarrow f(x)$ in Y.

i.e, claim that for every neighbourhood V of $f(x)$ there exist a positive integer N such that $f(x_i) \in V \quad \forall i \geq N$.

So, let V is a neighbourhood of $f(x)$

$\Rightarrow x \in f^{-1}(V)$.

Given $\{x_n\} \rightarrow x$ in X.

$\therefore$ For the neighbourhood $f^{-1}(V)$ of $x$ there exists a +ve integer N such that $x_i \in f^{-1}(V) \quad \forall i \geq N$.

$\Rightarrow f(x_i) \in V \quad \forall i \geq N$.

ie, neighbourhood $V$ of $f(x)$ a +ve integer $N$ such that, $f(x_i) \in V, \forall i \geq N$.

$\Rightarrow \{f(x_n)\} \rightarrow f(x)$ (36)

Conversely,

Let $(x_n) \rightarrow x$ in $X$.

$\Rightarrow \{f(x_n)\} \rightarrow f(x)$ in $Y$. 24

Claim: $f: X \rightarrow Y$ is continuous.

In order to prove $f$ is continuous.

It is enough to prove $f(\bar{A}) \subset \overline{f(A)}$. $f(\bar{A}) \subset \overline{f(A)}$

Where $A$ is a subset of $X$.

Let $x \in \bar{A} \Rightarrow f(x) \in f(\bar{A}) \rightarrow (1)$

By sequence lemma,

$x \in \bar{A} \Rightarrow$ there exist a sequence $(x_n)$ of points of $A$ converging to $x$.

By assumption,

$(x_n) \rightarrow x \Rightarrow (f(x_n)) \rightarrow f(x)$ where $f(x_n)$ will be the sequence of points of $f(A)$.

Again by the sequence lemma,

$(f(x_n))$ is a sequence of points if $f(A)$ converges to $f(x)$.

$\Rightarrow f(x) \in \overline{f(A)} \rightarrow (2)$

From (1) & (2), we have

$f(\bar{A}) \subset \overline{f(A)}$

$\therefore f: X \rightarrow Y$ is continuous.

---

Thm: 10.4.

In addition, subtraction, multiplication are continuous function from $R \times R$ into $R$ - the quotient operation is a continuous function $R \times R$ into $R$.

pf:-

case i,

Let $f$ be a addition function

To prove,

addition is continuous.

Let $f: R\times R \to R$ Such that $f(x,y) = x+y$. (37)

the space Metric in $R\times R$ is given by,

$$\rho((x,y),(x_0,y_0)) = \max\{|x-x_0|, |y-y_0|\}$$

Now, Prove that, $f: R\times R \to R$ is Continuous.

Let $\varepsilon>0$ be given, Let $\delta < \varepsilon/2$.

Now, Let $\rho((x,y),(x_0,y_0)) < \delta$.

$\Rightarrow \rho((x,y),(x_0,y_0)) < \varepsilon/2$ $(\because \delta < \varepsilon/2)$

$\Rightarrow \max\{|x-x_0|, |y-y_0|\} < \varepsilon/2$

$\Rightarrow |x-x_0| < \varepsilon/2,\ |y-y_0| < \varepsilon/2 \longrightarrow (1)$

Now, $d(f(x,y), f(x_0,y_0)) = d(x+y, x_0+y_0)$

$= |x+y-(x_0+y_0)| = |x-x_0+y-y_0|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$ [by (1)].

By thm, Let $f: X\to Y$ and Let $X$ and $Y$ are Metrizable with Metric $d_X$ and $d_Y$ respectively. The Continuity of $f$ equivalent to the requirement that given $x\in X$ and given $\varepsilon>0$ there exist $\delta>0$ such that $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

We have, $f$ is Continuous.

Case ii,

Prove, Multiplication is true.

Let $f$ be a Multiplication function $f: R\times R \to R$ by $f(x,y) = xy$

Claim $f$ is Continuous.

Let $\varepsilon>0$ be given

Prove that, for every $\varepsilon>0$ there exist $\delta>0$ such that $\rho((x,y),(x_0,y_0)) < \delta \Rightarrow d(f(x,y), f(x_0,y_0)) < \varepsilon$

ie, to prove

$$d(f(x,y), f(x_0,y_0)) < \varepsilon.$$

Now, $d(f(x,y), f(x_0,y_0)) = d(xy, x_0y_0)$

$= |xy - x_0y_0|$

$= |xy - x_0y_0 + x_0y - x_0y + xy_0 - xy_0 + x_0y_0 - x_0y_0|$

$= |x(y-y_0) - x_0(y-y_0) + x_0(y-y_0) + y_0(x-x_0)|$

(38)

$$= |(x-x_0)(y-y_0) + x_0(y-y_0) + y_0(x-x_0)|$$
$$\leq |(x-x_0)(y-y_0)| + |x_0||y-y_0| + |y_0||x-x_0|$$
$$\leq |(x-x_0)(y-y_0)| + (|x_0|+|y_0|+1)|y-y_0| + (|x_0|+|y_0|+1)|x-x_0| \rightarrow (2)$$

Choose $\delta = \min\left(\frac{\varepsilon}{(|x_0|+|y_0|+1)3}, \sqrt{\varepsilon/3}\right)$ $(\because |x_0| < |x_0|+|y_0|+1)$

Since $\rho((x,y),(x_0,y_0)) < \delta$.

Max $\{|x-x_0|, |y-y_0|\} < \delta$.

$\Rightarrow |x-x_0| < \delta$, $|y-y_0| < \delta$.

$\delta < \frac{\varepsilon}{3(|x_0|+|y_0|+1)}$ also $\varepsilon < \sqrt{\varepsilon/3}$

Now eqn. (2) becomes.

$$d(f(x,y), f(x_0,y_0)) < \sqrt{\varepsilon/3}\sqrt{\varepsilon/3} + \frac{|x_0|+|y_0|+1}{3(|x_0|+|y_0|+1)}\varepsilon + \frac{|x_0|+|y_0|+1}{3(|x_0|+|y_0|+1)}\varepsilon$$
$$= \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$
$$d(f(x,y), f(x_0,y_0)) < \varepsilon$$

$f$ is continuous.

Case iii)

To prove the reciprocal function is continuous.

Consider $f: R\times R \rightarrow \{x_0\} \rightarrow R$ by $f(x) = \frac{1}{x}$.

Claim $f$ is continuous.

It is sufficient to prove that given $\varepsilon > 0$ there exist $\delta > 0$ such that $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$.

ie, to prove that $|x-x_0| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon$.

Now, $\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x x_0}\right| = \left|\frac{x - x_0}{x x_0}\right| \rightarrow (2)$

Choose $\delta = \min\left\{\frac{|x_0|}{2}, \frac{\varepsilon x_0^2}{2}\right\}$

Let $|x-x_0| < \delta \Rightarrow |x-x_0| < \frac{|x_0|}{2}$.

$$\Rightarrow -\frac{|x_0|}{2} < x - x_0 < \frac{|x_0|}{2}.$$

Multiplying by $|x_0|$, we get

$$-\frac{|x_0|^2}{2} < x|x_0| - |x_0|x_0 < \frac{|x_0|^2}{2}$$
$$-\frac{|x_0|^2}{2} < x x_0 - x_0^2 < \frac{|x_0|^2}{2} \quad [\because |x_0| = \sqrt{x_0^2} = x_0]$$

ie, $\frac{-x_0^2}{2} < x x_0 - x_0^2$

$-\frac{x_0^2}{2} + x_0^2 < x x_0$

$\Rightarrow \frac{x_0^2}{2} < x x_0$ ie, $x x_0 > \frac{x_0^2}{2}$.

ie $\left|\frac{1}{x x_0}\right| < \left(\frac{2}{x_0^2}\right)$ ie, $\left|\frac{1}{x x_0}\right| < \frac{2}{x_0^2}$.

eqn. (2) becomes

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon \frac{x_0^2}{2} \cdot \frac{2}{x_0^2} \leq \varepsilon.$$

$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$.

$\therefore f$ is continuous.

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Case iv,

claim subtraction function is continuous.

Let $f$ be the subtraction function.

ie, $f: R \times R \to R$ by $f(x,y) = x - y = x + (-y)$.

Since addition function is continuous it follows that subtraction is also continuous.

Case v,

claim the quotient function is continuous.

Define $f: R \times R \to \{x_0\} \to R$ by $f(x,y) = \frac{x}{y} = x \cdot \frac{1}{y}$.

We have proved the multiplication & reciprocal are continuous.

Hence $f$ is continuous.

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Thm : 10.5

If $X$ is a topological space if $f, g : X \to R$ are continuous function, then $f+g$, $f-g$, $fg$ are continuous if $g(x) \neq 0 \ \forall x$. Then $f/g$ is continuous

Pf:-

Given $f: X \to R$ & $g: X \to R$ are continuous function.

Define a function $h: X \to R \times R$ by $h(x) = (f(x), g(x))$

Since the component $f$ and $g$ are continuous.

$\therefore h$ is continuous.

claim $f+g$ is continuous.

Now, $(f+g)(x) = f(x) + g(x)$

$= (f \circ h)(x)$ $\quad \left[ (f \circ h)(x) = f(h(x)) = f(f(x), g(x)) = f(x) + g(x) \right]$

Since the composition of continuous function is continuous ($\because f(x,y) = x+y$) & h is continuous.

$\Rightarrow f+g, f-g$ & $fg$ are continuous.

Claim, $f/g$ is continuous provided $g(x) \neq 0$.

Now, $(f/g)(x) = \dfrac{f(x)}{g(x)} = f(x) \dfrac{1}{g(x)}$.

Since $g(x)$ is continuous & reciprocal of continuous function is continuous.

$\Rightarrow \dfrac{1}{g(x)}$ is continuous.

Also, $f(x)$ is continuous. $\dfrac{1}{g(x)}$ is continuous and product of continuous function is continuous.

$\Rightarrow f/g$ is continuous.

Thm: 10.6

Uniform limit theorem

Let $f_n : X \to Y$ be a sequence of continuous function from the topological space X, to the metric space Y if $\{f_n\}$ converges uniformly to f, then f is continuous.

Pf:

Given $(f_n)$ be a sequence of continuous function

Where, $f_n : X \to Y$. Where Y is a metric space and $(f_n)$ converges uniformly to f.

Claim $f : X \to Y$ is also continuous.

For that, let V be an open in Y such that $x_0 \in f^{-1}(V)$
$\Rightarrow f(x_0) \in V$.

Now claim $f^{-1}(V)$ is open in X.

Now claim there exist a neighbourhood U of $x_0$ such that $x_0 \in U \subset f^{-1}(V)$

ie, claim that,

$f(U) \subset V$ where U is a neighbourhood of $x_0$.

Let $f(x_0) = y_0$.

Now, V is open in the metric space Y and $y_0 \in V$.

$\Rightarrow$ for given $\varepsilon > 0$ there exist a ball $B_d(y_0, \varepsilon)$ such that $B_d(y_0, \varepsilon) \subset V$. $\rightarrow (1)$

By defn. of Uniform converges (4) 29

$(f_n) \to f$ Uniformly.

for given $\varepsilon > 0$ there exist a +ve number $d(f_n(x), f(x)) < \varepsilon/4$.

The inequality is true $\forall x \in X$. $\forall n \geq n_0, x \in X \to (2)$

Put $x = x_0$ in (2), $d(f_n(x_0), f(x_0)) < \varepsilon/4 \to (3)$

Also given $(f_n)$ be a sequence of a continuous function.

$f_N : X \to Y$ is also continuous (Take $n = N$)

for every open ball $B_d(f_N(x_0), \varepsilon/2)$ in $Y$. its preimage $f_N^{-1}(B_d(f_N(x_0), \varepsilon/2))$ is open in $X$.

$\Rightarrow$ there exist a neighbourhood $U$ of $f_N(x_0)$ such that $x_0 \in U \subset f_N^{-1}(B_d(f_N(x_0), \varepsilon/2))$

$\Rightarrow f_N(U) \subset B_d(f_N(x_0), \varepsilon/2)$

So if $x \in U$. Then $f_N(x) \in f_N(U) \subset B_d(f_N(x_0), \varepsilon/2)$

$\therefore f_N(x) \in B_d(f_N(x_0), \varepsilon/2)$

$\Rightarrow d(f_N(x), f_N(x_0)) < \varepsilon/2 \longrightarrow (4)$

Now, $d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))$

$< \varepsilon/4 + \varepsilon/2 + \varepsilon/4$ (Take $n = N$)

$= \varepsilon$.

$\Rightarrow f(x) \in B_d(f(x_0), \varepsilon)$

$\Rightarrow f(U) \subset B_d(f(x_0), \varepsilon) \subset V$ [by (1)] [$\because x \in U \Rightarrow f(x) \in f(U)$]

$f(U) \subset V$ $\therefore U \subset f^{-1}(V)$.

$f$ is continuous $\Rightarrow f^{-1}(V)$ is open in $Y$.

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## UNIT - III

## CONNECTEDNESS AND COMPACTNESS

Criterian for the Connectedness :-

A space $X$ is connected iff only ~~the~~ that ~~there~~ are both open and closed subsets of $X$ on $X$ are the empty set and $X$ itself.

Pf :-

Let $A$ be a non-empty subset of $X$ which is both open and closed.

Let $X$ be connected.

# TOPOLOGY

## UNIT 3

Unit-III

connected spaces - connected sets in the real line - components and path components - local connectedness.

By defn. of Uniform Converges (4) 29

$(f_n) \to f$ Uniformly.

for given $\varepsilon > 0$ there exist a +ve number $d(f_n(x), f(x)) < \varepsilon/4$.

The inequality is true $\forall x \in X$. $\forall n \geq n_0, x \in X \to (1)$

Put $x = x_0$ in (2), $d(f_n(x_0), f(x_0)) < \varepsilon/4 \to (3)$

Also given $(f_n)$ be a sequence of a continuous function.

$f_N : X \to Y$ is also continuous (Take $n = N$)

for every open ball $B_d(f_N(x_0), \varepsilon/2)$ in $Y$. its preimage $f_N^{-1}(B_d(f_N(x_0), \varepsilon/2))$ is open in $X$.

$\Rightarrow$ there exist a neighbourhood $U$ of $f_N(x_0)$ such that $x_0 \in U \subset f_N^{-1}(B_d(f_N(x_0), \varepsilon/2))$

$\Rightarrow f_N(U) \subset B_d(f_N(x_0), \varepsilon/2)$

So if $x \in U$. Then $f_N(x) \in f_N(U) \subset B_d(f_N(x_0), \varepsilon/2)$

$\therefore f_N(x) \in B_d(f_N(x_0), \varepsilon/2)$

$\Rightarrow d(f_N(x), f_N(x_0)) < \varepsilon/2 \to (4)$

Now, $d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))$

$\leq \varepsilon/4 + \varepsilon/2 + \varepsilon/4$ (Take $n = N$)

$= \varepsilon$.

$\Rightarrow f(x) \in B_d(f(x_0), \varepsilon)$

$\Rightarrow f(U) \subset B_d(f(x_0), \varepsilon) \subset V$ [by (1)] [$\because x \in U \Rightarrow f(x) \in f(U)$]

$f(U) \subset V$ $\quad \because U \subset f^{-1}(V)$

$f$ is continuous $\Rightarrow f^{-1}(V)$ is open in $Y$.

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## UNIT - III

## CONNECTEDNESS AND COMPACTNESS

Criterian for the Connectedness :-

A space $X$ is connected iff only ~~the~~ subsets of $X$ that ~~there~~ are both open and closed on $X$ are the empty set and $X$ itself.

Pf :-

Let $A$ be a non-empty subset of $X$ which is both open and closed.

Let $X$ be connected.

Claim,

$A$ is either $\phi$ (or) $X$.

Let $U = A$.

Since $A$ is open. $U$ is an open set. Again let $V = X - A$. Since $A$ is closed. $X - A$ is open.

$\therefore V$ is also open.

$A$ is non-empty $\Rightarrow U$ is non-empty & $V = X - A$ is also non-empty.

$\therefore U$ & $V$ be a pair of non-empty open set in $X$ such that, $U \cap V = \phi$ & ~~UAV=~~ $U \cup V = X$.

$\therefore$ The $(U, V)$ is also seperation in $X$.

Which is a $\Rightarrow\Leftarrow$ to the assumption $X$ is connected.

$\therefore$ The only subsets of $X$ which are both open and closed in $X$ are $\phi$ & $X$ only.

Conversely,

Let the only set $X$ which both open and closed in $X$ be $\phi$ and $X$.

claim $X$ is connected.

Suppose, $X$ is not connected.

Then, there exist a seperation $(U, V)$ be pair of non-empty disjoint open subsets of $X$ such that $U \cup V = X$.

Consider, the non-empty set $U$ by the assumption it is open in $X$. $\longrightarrow (1)$.

Again $U \cup V = X \Rightarrow U = X - V$.

Now,

$V$ is open in $X \Rightarrow X - V$ is closed in $X$

$\Rightarrow U$ is closed in $X$. $\longrightarrow (2)$

By (1) & (2), we have

The non-empty set $U$ is both open and closed in $X$ which is a $\Rightarrow\Leftarrow$ to our assumption.

$\therefore X$ is connected.

---

Seperation in a subspace:

Lemma 1.1

Let $Y$ be a subspace of $X$. Then a separation of $Y$ in a pair of disjoint non-empty sets $A$ & $B$ whose union is $Y$ neither of its contains a limit points of the other. The subspace $Y$ is connected if there exist no seperation of $Y$.

Pf:

Let the pair $(A,B)$ be a separation of $Y$
i.e., $(A,B)$ are non-empty open subsets of $Y$.
Such that $A \cap B = \phi$ & $A \cup B = Y$.
Claim,
A and B are disjoint subsets of $Y$ such that neither of which contains the limit point of the other.
By assumption,
$A$ and $B$ are open in $Y$.
Again $A = Y - B$ & $B$ is open $\Rightarrow$ $A$ is closed in $Y$.
Thus $A$ is both open & closed in $Y$
Consider, the closure of $A$ in $Y$. i.e. the set $\bar{A} \cap Y$ where $\bar{A}$ is the closure of $A$ in $X$.
Now,
$A$ is closed in $Y \Rightarrow A =$ its closure in $Y$.
$\Rightarrow A = \bar{A} \cap Y$.

Consider $A \cap B = (\bar{A} \cap Y) \cap B$
$= \bar{A} \cap (Y \cap B) = \bar{A} \cap B$.
$A \cap B = \phi \Rightarrow \bar{A} \cap B = \phi$

$\Rightarrow$ B cannot contain the limit point of A also
IIly we can prove A also doesnot contain the limit of B.
Then neither of which contains the limit point of the other.
Conversely,
Let A and B are disjoint nonempty subsets of $Y$.
Where $Y$ neither of which contains a limit point of the other.
i.e, A and B are non-empty set such that $A \cap B = \phi$
$A \cup B = Y$, $A' \cap B = \phi$ & $A \cap B' = \phi$
Claim the pair $(A,B)$ is a separation of $Y$.
For that it is enough to prove A and B are open in $Y$.
First claim,
$\bar{A} \cap Y = A$, $\bar{B} \cap Y = B$.

Now, $\bar{A} \cap Y = (A \cup A') \cap (A \cup B)$
$= (A \cap (A \cup B)) \cup (A' \cap (A \cup B)) = (A \cap A) \cup (A \cap B) \cup (A' \cap A) \cup (A' \cap B)$
$= (A \cup (A' \cap A)) \cup (A' \cap B) = A \cup (A' \cap A) = A \cup \underbrace{(A' \cap A)}_{= A}$.

$\bar{A} \cap Y = A$ (∵ $A' \cap A \subset A$)

similarly, $\bar{B} \cap Y = B$

$\bar{A} \cap Y = A \Rightarrow A$ is closed in $Y$

$\Rightarrow Y - A$ is open in $Y$ (∵ $Y = A \cup B$)

$\Rightarrow B$ is open in $Y$.

$|||^{ly}$, $\bar{B} \cap Y = B \Rightarrow B$ is closed in $Y$

$\Rightarrow A$ is open in $Y$.

A is a pair of nonempty disjoint open subsets of $Y$ such that $Y = A \cup B$.

∴ The pair $(A, B)$ is a separation of $Y$.

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※) Equation of connected space:-

Let $X$ be a two product space in the discrete topology $Y$. then there is a no seperation of $X$. So $X$ is connected.

pf:

Let $X = \{a, b\}$

The topology $J = \{\phi, X\}$ (indiscrete topology)

Eventhough the pair of subsets $\{a\}, \{b\}$ are non-empty disjoint are not open.

∴ $\{a\}, \{b\}$ cannot form a separation on $X$.

∴ $X$ is connected,

Eqn. for a separation (or) disconnected :-

Let $Y = [-1, 0) \cup (0, 1]$ is subspace in $R$.

$[-1, 0) = (a, 0) \cap ([-1, 0) \cup (0, 1])$

$(a, 0)$ is open in $R$ & $a < -1$

$\Rightarrow (a, 0) \cap ([-1, 0) \cup (0, 1])$ is open in $Y$.

$|||^{ly}$ $(0, 1]$ is also open in $Y$

∴ $([-1, 0), (0, 1])$ is a pair of non-empty disjoint open subsets of $Y$ such that $[-1, 0) \cup (0, 1] = Y$

∴ They form a seperation of $Y$.

Lemma 1.2

If the set $C$ & $D$, form a Separation of $X$ and if $Y$ is a connected subsets of $X$. Then $Y$ lies entirely with in either $C$ or $D$.

Pf:

Given C and D form a separation of X.

$\therefore$ C, D are non-empty disjoint open subsets of X such that $C \cup D = X$.

Since, C, D are open in $X \Rightarrow C \cap Y, D \cap Y$ are open in Y.

Also, $(C \cap Y) \cap (D \cap Y) = (C \cap D) \cap (Y \cap Y) = \phi \cap Y$

$(C \cap Y) \cap (D \cap Y) = \phi$ and $(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap (Y \cup Y)$

$= X \cap Y$

$(C \cap Y) \overset{\cup}{\cap} (D \cap Y) = \phi \quad X \cap Y = Y.$

So the two sets $C \cap Y$ and $D \cap Y$ are disjoint open sets such that their union is Y.

If they were both non-empty they would give a separation of Y.

$\Rightarrow$ Y is disconnected.

Which is a $\Rightarrow\Leftarrow$ to assumption Y is connected

$\therefore$ Either $C \cap Y = \phi$ (or) $D \cap Y = \phi$

If $C \cap Y = \phi$ then $(C \cap Y) \cup (D \cap Y) = Y$

$\Rightarrow \phi \cup (D \cap Y) = Y$

$\Rightarrow D \cap Y = Y$

$\Rightarrow Y \subset D$.

llly if $D \cap Y = \phi$, then $(C \cap Y) \cup (D \cap Y) = Y$

$(C \cap Y) \cup \phi = Y$

$\Rightarrow C \cap Y = Y$

$\Rightarrow Y \subset C$.

---

(X) Thm 1.3

The union of a collection of connected sets that have a point in common is connected.

Pf:

Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subsets of X which have a common point.

(i.e) $\bigcap_{\alpha \in J} A_\alpha \neq \phi$

$P \in \bigcap_{\alpha \in J} A_\alpha$.

Claim $Y = \bigcup_{\alpha \in J} A_\alpha$ is connected. Suppose Y is not connected.

Thus there exist a separation (C, D) a pair of non-empty disjoint open subsets of Y such that $C \cup D = Y$.

$p \in \bigcap_{\alpha \in J} A_\alpha \Rightarrow p \in A_\alpha \ \forall \alpha.$

$\Rightarrow p \in \bigcup_{\alpha \in J} A_\alpha.$

$\Rightarrow p \in C \cup D$

$\Rightarrow p \in C$ (or) $p \in D$.

Let $p \in C$

WKT $A_\alpha$ is connected subset of $Y$ and $(C,D)$ is a separation of $Y$.

By lemma 1.2, $A_\alpha$ lies entirely with in $C$ (or) with in $D$.

ie, $A_\alpha \subset C$ (or) $A_\alpha \subset D$.

$A_\alpha \subset D$ is not possible.

Because $A_\alpha$ contains a point $p$ of $C$.

$\therefore A_\alpha \subset C \ \forall \alpha$.

$\Rightarrow \bigcup_{\alpha \in J} A_\alpha \subset C$.

But, By the assumption, $Y = C \cup D$

$\therefore C \subset Y$.

Which is $\Rightarrow\Leftarrow$ $(C,D)$ is a pair of non-empty set of $Y$.

$\therefore Y$ is connected.

---

Thm: 1.4

Let $A$ be a connected subset of $X$. if $A \subset B \subset \bar{A}$. Then $B$ is also connected.

Pf:

Given $A$ is connected and $A \subset B \subset \bar{A}$

Claim $B$ is connected.

Suppose $B$ is not connected.

Then there exist a separation $(C,D)$ is pair of non-empty disjoint open subset of $B$ such that $B = C \cup D$.

Now, $A$ is a connected subset of $B$ & $C, D$ is separation to $B \Rightarrow$ either $A \subset C$ (or) $A \subset D$. By Lemma:

Suppose $A \subset C \Rightarrow \bar{A} \subset \bar{C}$

$C$ & $D$ are separation to $B$.

$\Rightarrow \bar{C} \cap D = \phi \quad (C \subset \bar{C})$

$\Rightarrow \bar{A} \cap D = \phi \quad (\because \bar{A} \subset \bar{C})$

$\Rightarrow B \cap D = \phi \quad (\because B \subset \bar{A})$

If the sets $C$ and $D$ form a separation of $X$ and if $Y$ is connected subspace of $X$, then $Y$ lies entirely within either $C$ or $D$.

$\Rightarrow (C \cup D) \cap D = \phi \quad (\because B = C \cup B)$

$\Rightarrow (C \cap D) \cup (D \cap D) = \phi$

$\phi \cup D = \phi \qquad (\because C \cap D = \phi)$

$\Rightarrow D = \phi$

Which is $\Rightarrow \Leftarrow$ to $D$ is a non-empty subset of $B$.

$\therefore B$ is connected.

Thm 1.5

(*) The image of a connected space under continuous map is connected.

Pf:

Let $f: X \to Y$ be a continuous map and Let $X$ be connected.

claim $Z = f(X)$ is connected.

Consider the restriction map $g: X \to Z$.

W.K.T, the restriction map is continuous map.

claim $Z$ is connected

Suppose $Z$ is not connected.

Then there exist a separation $(A, B)$ a pair of disjoint non-empty open sets $Z$ such that $Z = A \cup B$.

W.K.T, $g^{-1}(Z) = X$

$g^{-1}(A \cup B) = X$

ie, $g^{-1}(A) \cup g^{-1}(B) = X \to (1)$

Again $g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B)$

$g^{-1}(A) \cap g^{-1}(B) = g^{-1}(\phi) = \phi \to (2)$

A and B are open in $Z \Rightarrow g^{-1}(A)$ and $g^{-1}(B)$ are open in $X$.

Because $g$ is continuous function $\to (3)$

More over $A, B \neq \phi$.

$\Rightarrow g^{-1}(A), g^{-1}(B) \neq \phi \to (4)$

By (1), (2), (3), (4).

$(g^{-1}(A), g^{-1}(B))$ is a pair of non-empty disjoint open subset of $X$ such that $X = g^{-1}(A) \cup g^{-1}(B)$

$(g^{-1}(A), g^{-1}(B))$ is a separation of $X$.

Which is $\Rightarrow \Leftarrow$ to $X$ is connected.

$\therefore Z$ is connected

ie, $Z = f(X)$ is connected.

$\therefore$ Continuous image of a connected space is connected 36

Thm 1.6. [Cartesian Product of connected space is connected]

The Cartesian Product of two connected space is connected (or) A finite Cartesian product of connected space is connected.

pf:

step i,

First let us proved the theorem (By inspection hypothesis) for $n=2$.

Claim $X, Y$ is connected $\Rightarrow X \times Y$ is connected.

Consider a "base point" $a \times b$ in the product $X \times Y$.

Consider, the horizontal slice $X \times b$.

Consider the projection map $\pi_1 : X \times b \to X$ defined by $\pi_1(x, b) = x$.

Then $\pi_1^{-1} : X \to X \times b$ is continuous map because $\pi_1$ is homomorphism.

$\therefore (X \times b)$ is a continuous image of $X$.

Hence $X$ is connected $\Rightarrow X \times b$ is also connected.

llly, consider the vertical slice $x \times Y$.

Consider also the projection map $\pi_2 : x \times Y \to Y$ be defining $\pi_2(x, y) = y$.

Then $\pi_2^{-1} : Y \to x \times Y$ is continuous mapping (by thm 1.5)

$\therefore \pi_2$ is ~~bi~~-continuous.

The slice $x \times Y$ is continuous image of the connected space $Y$.

$x \times Y$ is also connected.

Then consider the 'T-shaped' space.

$T_x = (X \times b) \cup (x \times Y)$

Here $T_x$ is the union of two connected sets that have the common point $x \times b$.

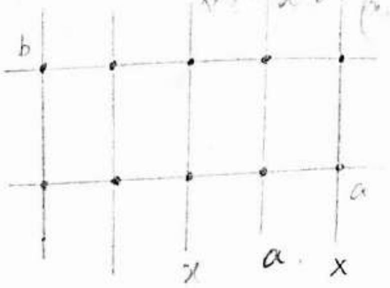
$\therefore T_x$ is also connected.

Now, from the union $\bigcup_{x \in X} T_x$ of all T shaped spaces.

This ~~Uniform~~ Union is connected. because $\cap T_x$ has the common point $a \times b$ (base element).

$X \times Y = \bigcup_{x \in X} T_x$ is Connected Space.

$\Rightarrow X \times Y$ is also Connected space.




Step ii)

Claim, Cartesian product of finite number Connected space is Connected.

Let $X = X_1 \times X_2 \times X_3 \times \dots \times X_n$. Where each $X_i$ is Connected.

Claim $X$ is Connected,

For this claim we shall follow the method of induction on $n$.

Let $n=1$, then $X = X_1$, which is obviously connected.

Let $n=2$, then $X = X_1 \times X_2$ which is also Connected (by step i)

Assume that,

the thm. is true for $(n-1)$.

ie, $X_1 \times X_2 \times \dots \times X_n = (X_1 \times X_2 \times \dots \times X_{n-1}) \times X_n$. which is the Cartesian product of two connected space. By step i, X is Connected, finite product space is also connected.

Step iii)

To prove an arbitrary product of connected space is connected.

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of connected space.

Let $X = \prod_{\alpha \in J} X_\alpha$.

Claim, $X$ is Connected.

Choose a base point $\bar{b} = (b_\alpha)_{\alpha \in J}$ for $X$.

Given any finite set $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ induces of $J$.

Consider the space $X(\alpha_1, \alpha_2 \dots \alpha_n)$

Obviously $X(\alpha_1, \alpha_2 \dots \alpha_n)$ is a subspace of $X$ & it consists of all points $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = b_\alpha$ for $\alpha \neq \alpha_1, \alpha_2 \dots \alpha_n$.

Claim, $X(\alpha_1, \alpha_2 \dots \alpha_n)$ is homeomorphic with finite product $\prod_{i=1}^{n} X_i$.

So, Consider the map $f : X_{\alpha_1} \times X_{\alpha_2} \times \dots \times X_{\alpha_n} \to X(\alpha_1, \alpha_2 \dots \alpha_n)$ defined by

$f(x_{\alpha_1}, x_{\alpha_2} \dots x_{\alpha_n}) = (y_\alpha)_{\alpha \in J}$ where $y_\alpha = x_\alpha$.

for $\alpha = \alpha_1, \alpha_2 \dots \alpha_n$ & $y_\alpha = b_\alpha$ for other value of $\alpha$.

Clearly, $f$ is a bijection.
Claim, $f$ is a continuous map.
$\therefore$ $f$ is a bijection if $f$ is open.
Then $f$ is also continuous, so it is enough to prove $f$ is an open map.
Consider,
an open set (basis element) namely $U_{\alpha_1} \times U_{\alpha_2} \times \dots U_{\alpha_n}$ for the product space $X_{\alpha_1} \times X_{\alpha_2} \times \dots X_{\alpha_n}$.
Now, $f(U_{\alpha_1} \times U_{\alpha_2} \times \dots U_{\alpha_n}) = (U_\alpha)_{\alpha \in J}$ such that $V_{\alpha_i}^{U_i} = U_\alpha$ for $\alpha = \alpha_1 \dots \alpha_n$ & $U_\alpha = B_\alpha$ for $\alpha \neq \alpha_1, \alpha_2 \dots \alpha_n$ (for the value of $\alpha$)
But $(U_\alpha)_{\alpha \in J}$ is the general base element for the subspace topology of the spaces $X(\alpha_1 \dots \alpha_n)$.
$\therefore$ $f$ takes an open set to an open set from $(X_{\alpha_1} \times \dots \times X_{\alpha_n})$.
$\therefore$ $f$ is open.
Since $f$ is a bijection and open.
$\therefore$ $f$ is also continuous.
$\therefore$ $X(\alpha_1 \dots \alpha_n)$ is continuous image of the connected spaces.
$X_{\alpha_1} \times \dots \times X_{\alpha_n}$.
(or) $\prod_{i=1}^{n} X_i$.

$\therefore$ $X(\alpha_1 \dots \alpha_n)$ is also connected.
($\because$ continuous image of connected space is connected).

Let $Y = \bigcup (X(\alpha_1, \alpha_2 \dots \alpha_n))$, where the union extends to all finite subsets $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ of $J$.
For every finite set of indices $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ $X\{\alpha_1, \dots \alpha_n\}$ of $J$ contains the base point $\bar{b} = (b_\alpha)_{\alpha \in J}$.
$\therefore$ $Y$ is the union of connected spaces having the common point $\bar{b} = (b_\alpha)_{\alpha \in J}$.
$\therefore$ $Y$ is also connected. [by thm 1.3 the union of collection of connected set that have a common point is connected].

Step(v)

Claim $\bar{Y} = X$, We know $\bar{Y} \subset X$ $\{\because Y \subset X\}$

so it is enough to prove $X \subset \bar{Y}$

Let $y = (y_\alpha)_{\alpha \in J} \in X$.

Claim $(y_\alpha) \in \bar{Y}$

For that claim every neighbourhood $U$ of $(y_\alpha)$ intersects $Y$.

Let $U$ be an arbitrary neighbourhood of $(y_\alpha)$.

Then, $U = \prod_{\alpha \in J} U_\alpha$, where $U_\alpha$ is open in $X_\alpha$ & $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_1 \ldots \alpha_n$ Construct a point.

$$y_\alpha = \begin{cases} x_\alpha & \text{for } \alpha = \alpha_1 \ldots \alpha_n \\ b_\alpha & \forall \text{ other value of } \alpha \end{cases}$$

Clearly $(y_\alpha)_{\alpha \in J} \in X(\alpha_1 \ldots \alpha_n)$

[Follows from the construction of $X(\alpha_1, \alpha_2 \ldots \alpha_n)$]

which implies $(y_\alpha) \in Y$ $\quad \because Y = \cup [X(\alpha_1, \alpha_2 \ldots \alpha_n)]$.

ie, $y_\alpha = x_\alpha \in U_\alpha$ $\forall$ $\alpha = \alpha_1, \alpha_2 \ldots \alpha_n$ {by defn of $\alpha$}

$y_\alpha = b_\alpha \in X_\alpha$.

Also $(y_\alpha)_{\alpha \in J} \in U$ for other value of $\alpha$.

ie, $(Y \cap U) \neq \phi$

Every basis neighbourhood $U$ of $(y_\alpha)_{\alpha \in J}$ intersects $Y$.

$\therefore (y_\alpha)_{\alpha \in J}$ is a limit point of $Y$.

$\Rightarrow (y_\alpha) \in \bar{Y}$, $\alpha \in J$

$\therefore X \subset \bar{Y}$ (IIIly $\bar{Y} \subset X$)

Hence $X = \bar{Y}$

We know $Y$ is connected $\Rightarrow$ $\bar{Y}$ is connected

$\Rightarrow X$ is connected $(\because X = \bar{Y})$.

Section 3.2.

Connected Set in the real line

Defn:

Thm :: /

(X) Let L be a linear continuous in the ordered topology. Then L is connected & so is every interval and ray on L. 2

Pf:

Let Y be a subspace of L.

Then Y will be equal to L (or) Y will be an interval in L (or) Y will be a ray in L.

Obviously, Y is convex, because if we takes a, b are any two points of Y. Then the [a,b] is contained in the set Y.

ie, $a,b \in Y$ with $a<b \Rightarrow [a,b] \subset Y \longrightarrow (1)$

Claim Y is connected.

Suppose Y is not connected.

Then there exist a separation A, B $\ni$: A & B are a disjoint nonempty open subsets of Y such that $Y = A \cup B$.

Since $A, B \in Y$ and choose $a \in A, b \in B$. Then by (1). $[a,b] \subset Y$, ~~$A_0 \times B_0 \in [a,b]$~~

So, consider the sets $A_0 = A \cap [a,b]$, $B_0 = B \cap [a,b]$.

A, B are open in $Y \Rightarrow A_0, B_0$ are open in $[a,b]$ in the subspace topology.

Obviously, $A_0 \neq \phi$ & $B_0 \neq \phi$ such that $A_0 \cap B_0 = \phi$

and $A_0 \cup B_0 = [(A \cap [a,b]) \cup (B \cap [a,b])]$

$= (A \cup B) \cap [a,b]$

$= Y \cap [a,b] = [a,b]$

$A_0 \cup B_0 = [a,b] \longrightarrow (*)$

L has the lub property $\Rightarrow$ Every infinite subset of L has a lub.

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$A_0$ and $B_0$ have a l.u.b

Let $C = \text{lub } A_0$.

Claim $C \notin A_0$ and $C \notin B_0$.

Case i,

claim $C \notin B_0$.

Suppose $C \in B_0 \Rightarrow C \in B \cap [a,b]$

$\Rightarrow C \in B$ and $C \in [a,b]$

$\Rightarrow C \notin A$ $\{\because B_0 = B \cap [a,b]$

$\Rightarrow C \neq a$ $A_0 = A \cap [a,b]\}$

Either $C = b$ (or) $a < c < b$.

$B_0$ is open in $[a,b]$.

there exist $d \in [a,b]$ such that $(d,c] \in B_0$ where $(d,c]$ is the basis elements of other topology.

Subcase i,

$C = b$

$(d,b] = (d,c] \subset B_0$ $[\because C \in B \Rightarrow (d,c] \subset B$

$\Rightarrow (d,b] \subset B_0$ $\Rightarrow (d,c] \subset B \cap [a,b]$

Claim $d$ is a lub of $A_0$. $\Rightarrow (d,c] = B_0]$

Suppose $d$ is not a lub of $A_0$.

there exist $a_0 \in A_0$ such that $a_0 > d$.

$\Rightarrow a_0 \in (d,c] = (d,b]$.

$a_0 \in (d,b] \subset B_0 \Rightarrow a_0 \subset B_0 \Rightarrow a_0 \in B$

which is $\Rightarrow \Leftarrow$ to $A \cap B = \phi$

$d$ is a lub of $A_0$. Now $d < c$ and $d$ is a lub of $A_0$ gives a $\Rightarrow \Leftarrow$ to $C = \text{lub of } A_0$. $\therefore C \neq b$.

Subcase ii,

Let $a < c < b$.

Now $C = \text{lub of } A_0$.

Every element of $A_0$ cannot be greater than $C$.

$\Rightarrow A_0$ cannot intersect $(C,b]$. [Because $C$ is an upper bound on $A_0$].

$\Rightarrow A_0 \cap (C,b] = \phi \rightarrow (1)$

Also, $A_0$ cannot intersect $(d,c]$.

Suppose $A_0 \cap (d,c] \neq \phi$

Then, there exist $a_0$ such that $a_0 \in A_0$ & $a_0 \in (d,c]$.

Now,

$a_0 \in (d, c] \subset B_0$

$\Rightarrow a_0 \in B_0$

$\Rightarrow a_0 \in B$.

Which is contradiction to $A \cap B = \phi$

$\therefore A_0 \cap (d, c] = \phi \rightarrow (2)$

from (1) and (2) We have

$A_0 \cap [(d, c] \cup (c, b]] = \phi$

$A_0 \cap (d, b] = \phi \rightarrow (3)$

claim $d$ = lub of $A_0$.

suppose not there exist $a_0 \in A_0$ such that $a_0 > d$

$\Rightarrow a_0 \in (d, b]$

$\Rightarrow a_0 \in A_0 \cap (d, b]$.

$\Rightarrow A_0 \cap (d, b] \neq \phi$

Which is Contradiction to (3)

$\therefore d$ is a lub of $A_0$.

Which is $\Rightarrow\Leftarrow$ to $c$ is a lub of $A_0$.

$\therefore a < c < b$ is not possible.

By subcase i) & ii), We have

$c \notin B_0$.

Case ii)

Claim $c \notin A_0$.

suppose $c \in A_0 \Rightarrow c \notin B_0$

$\Rightarrow c \neq b$.

$\therefore c = a$ (or) $a < c < b$.

since $A_0$ is open in $[a, b] \Rightarrow$ there exist some interval $[c, e)$ such that $[c, e) \subset A_0$.

since, $L$ is Linear Continuous and $c < e$

$\Rightarrow$ there exists $z \in [a, b]$ $\ni$: $c < z < e$.

$\Rightarrow z \in [c, e) \subset A_0$.

Thus We have an elements $z \in A_0$ $\ni$: $z > c$.

Which is a $\Rightarrow\Leftarrow$ $c$ = lub of $A_0$

$\therefore c \notin A_0$.

$\therefore C \notin A_0$ and $C \notin B_0 \Rightarrow C \notin A_0 \cap B_0$

$\Rightarrow C \notin A \cap B$.

But $C$ is a point of $Y = A \cup B$. 5

$c$ belongs to either $A$ or $B$.

$\therefore$ We get a $\Rightarrow\Leftarrow$ to the assumption.

$\therefore Y$ is connected.

---

Corollary :-

Real line $R$ is a linear continuous because it is a simply ordered set having two properties namely ($x<y$ there exist $z \in R \ni: x<z<y$ and has lub) property.

$\therefore R$ is also connected.

---

Intermediate Value theorem :-

Stmt :-

Let $f: X \to Y$ be a continuous map where $X$ is a connected space and $Y$ is an ordered set in the topology. If $a$ and $b$ are two points of $X$ and if $r$ is a point of $Y$ lying between $f(a) < f(b)$. Then there exist a point $c$ of $X \ni: f(c) = r$.

Pf :-

Given $f: X \to Y$ is continuous. Where $X$ is connected space.

$Y$ is an ordered set.

Let $a, b \in X$ such that $f(a), f(b) \in Y$

$r \in Y$ such that $f(a) < r < f(b)$.

Claim, there exist $c \in X$ such that $f(c) = r$.

Suppose there is no point $c$ of $X$ such that $f(c) = r \to (1)$

Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$

First $A$ and $B$ are non-empty.

Given $a, b \in X$

$\Rightarrow f(a), f(b) \in f(X) \subset Y$.

$r \in Y$ such that $f(a) < r < f(b)$.

$f(a) < r \Rightarrow f(a) \in A = f(X) \cap (-\alpha, r)$.

$f(b) > r \Rightarrow f(b) \in B = f(X) \cap (r, \alpha)$

$\therefore A \neq \phi$ & $B \neq \phi$. $\rightarrow (2)$

ii) Claim A and B are open in $f(X)$.

$(-\alpha, r)$ be a open ray in Y.

It is open in Y.

$\therefore f(X) \cap (-\alpha, r)$ is open in $f(X)$ subspace of Y.

Similarly,

$f(X) \cap (r, \alpha)$ is open in $f(X)$. since $(r, \alpha)$ is open in y

. Thus A and B are open in $f(X)$.

i.e, Obviously A and B are disjoint subsets of $f(X)$ $\hookrightarrow (3)$

i.e, claim $f(X) = A \cup B$.

By (1) for every $x \in X$, $f(x) < r$ (or) $f(x) > r$.

If $f(x) < r$.

then $f(x) \in A$. If $f(x) > r$ then $f(x) \in B$. Thus $\forall x$ in $X$ $\therefore f(x)$ will be either in A or B $\Rightarrow f(X) = A \cup B$

So by the above claim A, B are nonempty disjoint open subsets of $f(X)$ such that $f(X) = A \cup B$.

$\Rightarrow$ A, B form a separation of $f(X)$.

$\Rightarrow f(X)$ is not connected.

Which is $\Rightarrow \Leftarrow$ to the continuous image of a connected space is connected.

[Given X is connected].

$\therefore$ Our assumption (1) is wrong.

There exist a point $c \in X \ni: f(c) = r$

whenever $f(a) < r < f(b)$.

Thm:-

X is a path connected space $\Leftrightarrow$ X is connected but the converse is not true.

Pf:-

Let X be the path connected space.

there exist a continuous function $f: [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$ $\forall$ $x, y \in X$.

Claim, X is connected.

Suppose, $x$ is not connected

Then there exists a separation $(A,B)$ such that $X$ has a separation $A \cup B$.

W.K.T,

$[a,b]$ is connected and also,

W.K.T, Continuous image of a connected space is connected.

$\therefore f([a,b])$ is also connected.

Now,

$X$ has a separation and $f([a,b])$ is connected subset of $X$

$\Rightarrow f([a,b])$ is entirely either within $A$ (or) in $B$.

Suppose $f([a,b]) \subset A$.

Then $f(a) \in A$ and $f(b) \in A$.

$\therefore$ The path exists only between the points of $A$.

$||^{ly}$, if $F([a,b]) \subset B$.

then there exists no path in $X$ joining a points of $A$ to a points of $B$.

Which is $\Rightarrow\Leftarrow$ to the assumption that $X$ is the path connected.

$X$ is connected.

Conversely,

Connectedness need not imply path connected.

---

Thm :-

The components of $X$ are connected disjoint subspaces of $X$ whose union is $X$, such that each nonempty connected subspace of $X$ intersects only one of them.

Pf:

Being equivalence classes the components of $X$ are disjoint and their union is $X$.

Each connected subspace $A$ of $X$ intersects only of them.

For, if $A$ intersects the components $C_1$ and $C_2$ of $X$, say in points $x_1$ and $x_2$ respectively.

Then $x_1 \sim x_2$.

By defn,

this cannot happen unless $C_1 = C_2$.

To show the components $C$ is connected, choose a point $x_0$ of $C$. For each point $x$ of $C$. W.K.T,

$x_0 \sim x$, so there is a connected subspace $A_x$ containing $x_0$ and $x$.

By the result just proved $A_x \subset C$.

$$\therefore C = \bigcup_{x \in C} A_x.$$

Since, the subspaces $A_x$ are connected and have the point $x_0$ in common their union is connected.

---

Thm :-

A space $X$ is locally connected iff for every open set $U$ of $X$, each component of $U$ is open in $X$.

Pf::

Suppose that $X$ is locally connected.

Let $U$ be an open set in $X$.

Let $C$ be a component of $U$.

If $x$ is a point of $C$, we can choose a connected neighbourhood $V$ of $X$ such that $V \subset U$.

Since,

$V$ is connected, it must lie entirely in the component $C$ of $U$.

$\therefore C$ is open in $X$.

Conversely,

Suppose that components of open sets in $X$ are open.

Given a point $x$ of $X$ and a neighbourhood $U$ of $x$.

Let $C$ be the components of $U$ containing $x$.

Now $C$ is connected.

since it is open in $x$ by hypothesis,
$x$ is locally connected at $x$.

Thm :-

(x.) If $X$ is a topological space, each path component of $X$ lies in a component of $X$. If $X$ is locally path connected, then the components and the path components of $X$ are the same.

pf:

Let $C$ be a component of $X$.
Let $x$ be a point of $C$.
Let $P$ be the path component of $X$ containing $x$.
Since $P$ is connected $P \subset C$.
We wish to show that if $X$ is locally path connected, $P = C$.
Suppose that ~~fec~~ $P \subsetneq C$
Let $Q$ denote the union of all the path components of $X$ that are different from $P$ and intersect $C$. each of them necessarily lies in $C$.
So that $C = P \cup Q$.

Because $X$ is locally path connected each path component of $X$ is open in $X$.
$\therefore P$ [Which is a path component] and $Q$ [Which is a union of path components] are open in $X$.
So they constitute a separation of $C$.
This contradicts the fact that $C$ is connected.

## Unit-IV

Thm :-

(i) Let $Y$ be a subspace of $X$. Then $Y$ is compact iff every covering of $Y$ by sets open in $X$ contains a finite subcollection covering $Y$.

pf:

Assume that, $Y$ is compact.
Let $\mathcal{A}$ be a covering of $Y$ by sets open in $X$.

# TOPOLOGY

## UNIT 4

Unit - IV

compact spaces - compact sets in the real line - limit point compactness.

since it is open in $X$ by hypothesis,
$X$ is locally connected at $x$.

Thm :-

(x.) If $X$ is a topological space, each path component of $X$ lies in a component of $X$. If $X$ is locally path connected, then the components and the path components of $X$ are the same.

pf:

Let $C$ be a component of $X$.
Let $x$ be a point of $C$.
Let $P$ be the path component of $X$ containing $x$.
Since $P$ is connected $P \subset C$.
We wish to show that if $X$ is locally path connected, $P = C$.
Suppose that ~~Pc~~ $P \subsetneq C$
Let $Q$ denote the union of all the path components of $X$ that are different from $P$ and intersect $C$. Each of them necessarily lies in $C$.
So that $C = P \cup Q$.

Because $X$ is locally path connected each path component of $X$ is open in $X$.
$\therefore P$ [Which is a path component] and $Q$ [Which is a union of path components] are open in $X$.
So they constitute a separation of $C$.
This contradicts the fact that $C$ is connected.

## Unit-IV

Thm :-

i) Let $Y$ be a subspace of $X$. Then $Y$ is compact iff every covering of $Y$ by sets open in $X$ contains a finite subcollection covering $Y$.

pf:

Assume that, $Y$ is compact.
Let $\mathcal{A}$ be a covering of $Y$ by sets open in $X$.

To prove,

$\mathcal{A}$ contains a finite subcollection of $Y$

$U \in \mathcal{A}$ be arbitrary.

Since $U$ is open in $X$.

$U \cap Y$ is open in $Y$.

$\{U \cap Y / U \in \mathcal{A}\}$ is covering of $Y$ by sets open in $Y$.

Now, $Y$ is compact.

$\therefore$ there exist a finite subfamily $\{U_1 \cap Y, U_2 \cap Y, \ldots U_k \cap Y\}$ which covers $Y$.

$$Y \subset (U_1 \cap Y) \cup \ldots \cup (U_k \cap Y)$$
$$= (U_1 \cup U_2 \cup \ldots \cup U_k) \cap Y.$$
$$Y \subseteq U_1 \cup \ldots \cup U_k.$$

$\{U_1, U_2 \ldots U_k\}$ is the required finite subfamily $\mathcal{A}$ that covers $Y$.

Conversely,

Assume that every covering of $Y$ by sets open in $X$ has a finite subcovering of $Y$.

To prove,

that $Y$ is compact.

Let $\mathcal{A}$ be a covering of $Y$ by sets open in $Y$.

To prove that, $\mathcal{A}$ has a finite subcollection and also cover $Y$.

Let $U \subset \mathcal{A}$ be arbitrary.

Since $U$ is open in $Y$.

$U = V_U \cap Y$ where $V_U$ is open in $X$.

Now, $U \subseteq V_U$.

$$Y = \bigcup_{U \in \mathcal{A}} U \subseteq \bigcup_{V \in \mathcal{A}} V_U.$$

$\{V_U / U \in \mathcal{A}\}$ is a covering of $Y$ by sets open in $X$.

By propering, there exist a finite subfamily $V_{U_1}, V_{U_2}, \ldots V_{U_K}$ which covers $Y$.

$$Y \subseteq V_{U_1} \cup V_{U_2} \cup \ldots \cup V_{U_K}.$$
$$Y \cap Y = (V_{U_1} \cup V_{U_2} \cup \ldots V_{U_K}) \cap Y.$$
$$Y = (V_{U_1} \cap Y) \cup (V_{U_2} \cap Y) \cup \ldots (V_{U_K} \cap Y)$$

$Y = U_1 \cup U_2 \cdots \cup U_K$

$\{U_1, U_2 \cdots U_K\}$ is the required subfamily of $\mathcal{A}$ that covers $Y$.

$\therefore Y$ is compact.

---

(ix) Thm 5.2

Every closed subset of a compact space is compact

Pf:

Let $X$ be a compact and $Y \subseteq X$ be closed in $X$.
To prove that, $Y$ is compact.

Let $\mathcal{A}$ be any open cover for $Y$ by sets open in $X$.

By above Lemma, $\mathcal{A}$ contains open subsets of $X$ whose union $\supseteq Y$.

ie, $\mathcal{A} = \{U_\alpha / U_\alpha$ is open in $X\}$ and $\cup U_\alpha = Y$.

Since $Y$ is closed in $X$.

$X - Y$ is open in $X$.

Consider,

$\mathcal{B} = \mathcal{A} \cup (X - Y)$ it is open cover for $X$.

But $X$ is compact.

there exist a finite subfamily $\mathcal{C}$ of $\mathcal{B}$ that cover $X$.

Suppose $\{X - Y\} \in \mathcal{C}$ drop that and the remaining of $\mathcal{C}$ is the required finite subfamily of $\mathcal{A}$ that covers $Y$.

$\therefore Y$ is compact.

---

Thm : 5.3.

(x) Every compact subset of a Hausdroff space is closed.

Pf:

Let $X$ be a hausdroff space.

Let $Y \subset X$ be compact.

To prove that, $Y$ is closed in $X$.

ie, to prove that, $X-Y$ is open.
Let $x_0 \in X-Y$ be arbitrary. 12
Let $y \in Y$ be arbitrary.
Clearly,

$x_0 \neq y$ and $x_0, y_0 \in X$.

Since $X$ is Hausdroff space.
$\Rightarrow$ there exist a neighbourhood $U_y$ of $x_0$ and $V_y$ of $y$ such that $U_y \cap V_y = \phi$.
ie, for every $y \in Y$ there exist an neighbourhood $V_y$ of $y$ and $U_y$ of $x_0$ such that $U_y \cap V_y = \phi$
Clearly,

$Y \subseteq \bigcup_{y \in Y} V_y$.

$\{V_y / y \in Y\}$ is an open cover for $Y$.
Now, $Y$ is compact.
$\Rightarrow$ There exists a finite subfamily $V_{y_1}, V_{y_2} \ldots V_{y_k}$ which also covers $Y$.
Now, Consider the corresponding neighbourhood $U_{y_1}, U_{y_2} \ldots U_{y_k}$ of $x_0$.

Let $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_k}$

Since, finite intersection of open sets is open.
We have,

$U$ is an open set containing $x_0$.
$\therefore U$ is a neighbourhood of $x_0$.

Now, $U_{y_i} \cap V_{y_i} = \phi \quad \forall i = 1, to\ k$.
Claim,

$U$ does not intersect $V_{y_1} \cup V_{y_2} \cup \ldots V_{y_k} = V$.

ie, Claim $U \cap V = \phi$

Where $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_k}$ and $V = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_k}$.

If $U \cap V \neq \phi$

Let $z \in U \cap V \Rightarrow z \in U$ and $z \in V$.

$z \in U \Rightarrow z \in U_i \ \forall i$ and
$z \in V \Rightarrow z \in V_i$ for some $i$

Let $z \in V_j$

$\because z \in U_j$ and $z \in V_j \Rightarrow z \in U_j \cap V_j$ where $U_j / V_j$ are neighbourhood of $x_0$ and $y_0$ of $X$. 13

Which is a $\Rightarrow \Leftarrow$ to $X$ is a hausdroff space.

$U \cap V = \phi$

$U \cap Y = \phi$ $[\because Y \subset V]$

$U \subset X - Y$.

ie, $X - Y$ contains a neighbourhood of each of its points.

$\therefore X - Y$ is open $\Rightarrow Y$ is closed.

---

Lemma 5.4

$Y$ is compact subset of a Hausdroff space $X$ and $x_0$ is not in $Y$. Then there exist disjoint open set $U$ and $V$ of $X$ containing $x_0$ and $Y$ respectively.

Pf:

Let $X$ be a hausdroff space.

Write the proof of the previous thm upto $U \cap V = \phi$

Hence, we have proved there exists disjoint open sets $U$ and $V$ such that $U$ contains $x_0$ and $V$ contains $Y$.

Thm 5.5.

The image of a compact space under a continuous map is compact.

Pf:

Let $f: X \to Y$ be continuous.

Let $X$ be compact space.

To prove, $f(X)$ is compact in $Y$.

Let $\mathcal{A} = \{U_\alpha / U_\alpha \subset f(X)\}$ be arbitrary covering of $f(X)$ by sets open in $Y$.

Consider, the collection $\{f^{-1}(U_\alpha) / U_\alpha \in \mathcal{A}\}$

for each $U_\alpha \in \mathcal{A}$, $f^{-1}(U_\alpha)$ is open in $X$.

($\because f$ is continuous)

$A$ covers $f(X) \Rightarrow f(X) \subseteq \left( \bigcup_{U_\alpha \in A} U_\alpha \right)$

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$\Rightarrow X \subseteq f^{-1}\left(\bigcup_{U_\alpha \in \mathcal{A}} U_\alpha\right)$

$\Rightarrow X \subseteq \bigcup_{U_\alpha \in \mathcal{A}} f^{-1}(U_\alpha)$

$\Rightarrow \{f^{-1}(U_\alpha) / U_\alpha \in \mathcal{A}\}$ is a covering of $X$ by sets open in $X$.

But $X$ is compact.

$\therefore$ there exist a finite subfamily $f^{-1}(U_1), f^{-1}(U_2), \ldots f^{-1}(U_K)$ covers $X$

ie, $X \subset f^{-1}(U_1) \cup f^{-1}(U_2) \cup \ldots f^{-1}(U_K)$.

$\Rightarrow f(X) \subset U_1 \cup U_2 \ldots \cup U_K$.

$\{U_1, U_2 \ldots U_K\}$ is the required subfamily of $\mathcal{A}$ which also covers $f(X)$

$\therefore f(X)$ is compact.

---

Thm 5.6

Let $f: X \to Y$ be a bijective continuous function if $X$ is compact and $Y$ is hausdroff space then $f$ is homeomorphism.

Pf:

Given $f: X \to Y$ is bijective and continuous.

To prove that,

$f$ is homeomorphism.

It is sufficient to prove that, $f^{-1}$ is continuous where $f^{-1}: Y \to X$.

Let $A$ be any closed subset of $X$.

It is required to prove that,

$(f^{-1})^{-1}(A) = f(A)$ is closed in $Y$.

[$\because$ A closed subset of a compact space is compact].

Since $A$ is closed in $X$.

$\Rightarrow A$ is compact in $X$. (by above thm)

Since $f(A)$ is compact in $Y$ and $Y$ is hausdroff space.

$\Rightarrow f(A)$ is closed

[$\because$ A is compact subset of a hausdroff space closed].

$\therefore f^{-1}$ is continuous.

$\therefore f$ is homeomorphism.

Thm 5.7.

The product of finitely many compact space is compact.

pf:

First we shall prove that, the product of two compact space is compact.

Then the thm, follows by induction for any finite product.

Let us assume $X$ and $Y$ are compact space.

Claim, $X \times Y$ is compact.

Step i,

Before proving this thm,

Let us prove the following lemma,

Proof of tube lemma The Lemma,

Let $x_0$ be a point of $X$. and $N$ is a open subset of $X \times Y$ containing the slice $x_0 \times Y$. Now prove that there is a neighbourhood $W$ of $x_0$ in $X$ such that $N$ contains $W \times Y$ entirely set.

pf:

Let us cover the slice $x_0 \times Y$ by basis element $\{U \times V\}$ lying in $N$.

Now,

the slice $x_0 \times Y$ is homeomorphic, to space $Y$ and $Y$ is compact.

$\Rightarrow x_0 \times Y$ is compact [$\because$ The compact image of a continuous space is continuous]

Hence the slice $x_0 \times Y$ be covered by finitely many basis element $U_1 \times V_1, U_2 \times V_2 \ldots U_n \times V_n$.

Let us assume each of the basis element $U_i \times V_i$ intersects $x_0 \times Y \rightarrow (*)$

Define $W = U_1 \cap U_2 \cap \ldots \cap U_n$.

Since $U_1, U_2 \ldots U_n$ are open.

$\Rightarrow W$ is open.

[∵ Finite intersection of open sets is open].

Also, the open set $W$ containing $x_0$ [because by assumption (*) each $U_i \times V_i$ in the compact cover intersects $x_0 \times Y$]. 16

$\Rightarrow x_0 \in V_i$ for each $i \Rightarrow x_0 \subset W$.

Next claim, the sets $\{U_i \times V_i\}$ which are taken as cover to the slice $x_0 \times Y$ is also covering the tuble $W \times Y$.

Let $x \times y \in W \times y$.

Consider, the point $x_0 \times y$ of the slice $x_0 \times Y$ having the same $Y$-co-ordinate of the point ~~xxy~~. $x \times y$.

Since the sets $\{U_i \times V_i\}$ is a cover to slice $x_0 \times Y$.

$x_0 \times Y \subset \cup (U_i \times V_i)$, $i = 1$ to $n$.

$\Rightarrow (x_0 \times y) \in x_0 \times Y \Rightarrow x_0 \times y \in \cup (U_i \times V_i)$

$\Rightarrow x_0 \times y \in U_i \times V_i$ for some $i$.

$\Rightarrow x_0 \in U_i$ and $y \in V_i$ for some $i$.

$\Rightarrow y \in V_i \rightarrow (1)$ for some $i$.

Again take $x \times y \in W \times Y \Rightarrow x \in W$

$\Rightarrow x \in U_1 \cap U_2 \dots \cap U_n$.

$\Rightarrow x \in U_i \rightarrow (2)$

(1) & (2), $x \times y \in U_i \times V_i$, for some $i$.

$\Rightarrow x \times y \in \cup (U_i \times V_i)$

$\Rightarrow W \times Y \subset \cup (U_i \times V_i)$

$\Rightarrow$ The sets $\{U_i \times V_i\}$ cover the tube $W \times Y$.

Now, each $U_i \times V_i \subset N \Rightarrow \cup (U_i \times V_i) \subset N$

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$\Rightarrow W \times Y \subset N$ [∵ $W \times y \subset (U_i \times V_i) \subset N$]

Then $W$ is the required open set containing $x_0$ such that the open set $N$ contains the tube $W \times Y$ entirely.

Step ii,

Claim, $X \times Y$ is compact.

Let $\mathcal{A}$ be an open covering of $X \times Y$.

Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may be covered by finitely many elements $A_1, A_2 \dots A_m$ of $\mathcal{A}$. 17

The union $N = A_1 \cup A_2 \cup \dots \cup A_n$ is a set containing $x_0 \times Y$ by step i,. the open set contains the tube $W \times Y$ about $x_0 \times Y$. Where $W$ is open in $X$.

Then $W \times Y$ is covered by finitely many elements of $\mathcal{A}$.

Thus, for each $x$ in $X$, we can choose a neighbourhood $W_x$ of $x$ such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

The collection of all the neighbourhood $W_x$ is open covering of $X$.

By compactness of $X$,

there exists a finite subcollection $\{W_1, W_2 \dots W_K\}$ covering $X$.

The union of all tubes $W_1 \times Y, W_2 \times Y \dots W_K \times Y$ is all of $X \times Y$.

Since each may be covered by finitely many elements of $\mathcal{A}$. So many $X \times Y$ be covered.

$\therefore X \times Y$ is compact

Step iii,

Finite product of compact space is compact.
ie, Assume $X_1 \times X_2 \times \dots \times X_n$ compact space.
and prove $X_1 \times X_2 \times \dots X_n$ are compact.
We can prove this result by induction on $n$.

Let $X = X_1 \times X_2 \times \dots \times X_n$ for $n = 1$,
then $X = X_1$

which is obviously compact.

Let $n = 2$, $X = X_1 \times X_2$ which is also compact for step(iii)
Assume the result is true for $n < k-1$.
ie, $X_1 \times X_2 \times \dots \times X_{k-1}$ is compact.

$\Rightarrow X_1 \times X_2 \times \dots \times X_{k-1}$ is also ~~Connecte~~ Compact.

Consider the space $X_1 \times X_2 \dots \times X_k$.

$X_1 \times X_2 \times \cdots \times X_K$ is Compact [By assumption and $X_K$ is compact].

$\Rightarrow X_1 \times X_2 \times \cdots \times X_K$ is compact.

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Thus the result is true for any $K$.

$\therefore$ finite product of a Compact space is Compact.

---

Lemma : 5.8

Tube Lemma

Consider the product space $X \times Y$. Where $Y$ is Compact. if $N$ is an open set of $X \times Y$ Containing the slice $x_0 \times y$ of $X \times Y$. then $N$ Contains some tube $W \times y$ about $x_0 \times Y$. Where $W$ is neighbourhood of $x_0$ in $X$.

pf:-

Step (1) in the previous thm.

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Thm 5.9.

Let $X$ be a topological space. Then $X$ is compact iff for every Collection $\mathcal{C}$ of Closed sets in $X$ having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

pf:-

Step i,

Let $\mathcal{A}$ be the Collection of subset of $X$.
Let $\mathcal{C} = \{X - A / A \in \mathcal{A}\}$

then We Will prove the following statement;

i) $\mathcal{A}$ is a Collection of open sets iff $\mathcal{C}$ is a Collection of Closed sets.

ii) The collection of $\mathcal{A}$ covers $X$ iff $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is empty.

iii) The finite subcollection $\{A_1, A_2, \ldots A_n\}$ of $\mathcal{A}$ covers $X$ iff the intersection of the Corresponding elements $C_i = X - A_i$ of $\mathcal{C}$ is empty.

pf:-

$A$ is open.

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Now, $\mathcal{C} = \{X - A / A \in \mathcal{A}\}$

$\therefore X - A$ is closed.

$\therefore \mathcal{C}$ is closed.

i) The first statement is trivial. because the Complement of a open set is Closed.

ii) Let $\mathcal{A} = \{A \subset X\}$ covers $X$.

Then, $\cup A_i = X$.

Claim,

$$\cap C_i = \phi, \ \cap C_i = \cap (X - A_i) = X - \cup A_i = X - X = \phi$$

$$\bigcap_{i=1}^{n} C_i = \phi$$

iii) Let the collection $\{A_1, A_2 \dots A_n\}$

$$X = \bigcup_{i=1}^{n} A_i$$

Claim, $\bigcap_{i=1}^{n} C_i = \phi$

$$\bigcap_{i=1}^{n} C_i = \bigcap_{i=1}^{n} X - A_i = X - \bigcup_{i=1}^{n} A_i = X - X = \phi$$

Proof of main thm,

Conversely,

Finite intersection is non-empty

$\overset{\text{not}}{\Rightarrow} \cap C = \phi$

To prove that, $X$ is compact.

ie, To prove, every open cover $\mathcal{A}$ of $X$ has a finite subcover covers to $X$.

This statement is equivalent to contra positive statement.

$\Rightarrow$ Given any collection $\mathcal{A}$ of open sets is no finite subcollection of $\mathcal{A}$ covers $X$.

Then $\mathcal{A}$ does not covers $X$.

Let, $\mathcal{A}$ be a collection of open sets in $X$.

Then, $\mathcal{C} = \{X - A / A \in \mathcal{A}\}$ be a collection of Closed sets in $X$ [by (i)].

Assume that no finite Collection of $\mathcal{A}$ covers $X$.

$\{$ie, $\cup A_i \neq X\}$

$\Rightarrow \bigcap_{i=1}^{n} C_i \neq \phi$ [by (iii)]

By statement (ii) $\bigcap_{C\in \mathcal{C}} \neq \phi$

$\Rightarrow \mathcal{A}$ does not cover $X$.

Thus no finite subcollection $A$ covers $X$

$\Rightarrow A$ itself doesnot covers $X$.

Which is $\Rightarrow \Leftarrow$

$\therefore X$ is compact.

Proof:

Let $X$ be compact

Claim, $\bigcap_{i=1}^{n} C_i = \phi \Rightarrow \bigcap_{i=1}^{n} C_i \neq \phi$

Let $A = \{A_\alpha / A_\alpha$ is open in $X\}$

Let $\cup A_\alpha = X$.

$\Rightarrow \mathcal{C} = \{X - A / (X-A)$ is closed $W_x\} \Rightarrow \cap C_i = \phi$

[Let $\mathcal{C}$ = {collection of closed subset of $X$} and $X$ is compact. Also,

Let $\bigcap_{i=1}^{n} \underset{(C\in\mathcal{C})}{C_i} \neq \phi \longrightarrow (*)$

Claim, $\bigcap_{C\in\mathcal{C}} C \neq \phi$

Given $\mathcal{C} = \{C / C$ is closed in $X\}$

Let $S = \{A = X - C / C \in \mathcal{C}\}$

To prove, $\bigcap_{C\in\mathcal{C}} C \neq \phi$

Suppose $\cap C \neq \phi$

By result (2), $S$ covers $X$.

But $X$ is compact.

Any finite subcollection $\{A_1, A_2 \ldots A_n\}$ of $S$ and also cover $X$. Again by there exist result (3).

Thm 6.1

Compact subspace in the real line.

Let $X$ be simply ordered set having the l.u.b property in the ordered topology each closed interval in $X$ is compact.

Pf:

Given X is a simply ordered set.
Let a<b in X.
claim, [a,b] is compact.
Let $\mathcal{A}$ be an open covering of [a,b] by ~~with~~ sets open in [a,b] under the subspace, topology [Which is same as the ordered topology.]
We will prove that, there exists a finite subcollection of $\mathcal{A}$ covering [a,b].
Before proving the main thm,
Let us prove the lemma,
Step i,

Let x be a point of [a,b] difference from b.
Then there is a point y>x in [a,b] such that the interval [x,y] can be covered by atmost two elements of A.
Case i,

Suppose x has an immediate successor in X.
Let y be this immediate successor.
Then the [x,y] consists of two points x and y.
So that it can be covered by atmost two elements of A.

case ii,

Suppose x has no immediate successor in X
Choose an element A of $\mathcal{A}$ containing x.
Because ~~Since~~ x≠b and A is open.
then A contains an interval of the form [x,c) for some c∈[a,b]
choose a point y∈(x,c) then [x,y] is covered by the single element A of $\mathcal{A}$.

**Step ii,**

Let C be the set of all points y>a of [a,b] such that the interval [a,y] can be covered by finitely many elements of $\mathcal{A}$.

ie, $\mathcal{C} = \{y \in [a,b]$ and $y > a / [a,y]$ can be covered by finitely many points of $\mathcal{A}\} \to (i)$,

Applying step i) to the case $x = a$.

We see that $[a,y]$ can be covered by atmost two elements.

We see that atleast one such y. So C is non empty

The collection $\mathcal{C}$ is non-empty.

By the property of lub $\mathcal{C}$ has the lub (say) c. Let C be the lub of the

$\therefore c = \text{lub } \mathcal{C}$ then $a < c \leq b$. Sol C, then $a < c \leq b$.

Step iii,

Claim, $c \in \mathcal{C}$

ie, To prove that, the interval $[a,c]$ can be covered by finitely many elements of $\mathcal{A}$.

$c \in A$

Choose an element A of $\mathcal{A}$ containing c.

ie, $c \in A$, since A is open.

then it contains an interval of the form $(d,c]$ for some d in $[a,b]$.

If c is not in $\mathcal{C}$ there must be a point z of $\mathcal{C}$ lying in the interval $(d,c)$ because otherwise d would be a smaller upper bound of $\mathcal{C}$ and less than c.

since z is in $\mathcal{C}$.

then the interval $[a,z]$ can be covered by finitely n elements of $\mathcal{A}$.

$(d,c] \subset A$

More that $[z,c]$ lies in the single element of A of $\mathcal{A}$.

Now, $[a,c] = [a,z] \cup [z,c]$ can be covered by n+1 elements of A.

$$\left[\ [z,c] \subset (d,c] \subset A \Rightarrow [z,c] \subset A\ \right].$$

Which is $\Rightarrow\Leftarrow$ to

$\therefore c \in \mathcal{C}$.

Step iv,

Claim, $c = b$.

Suppose $c \neq b$ then $c < b$.

Applying step i) to the point $x = c$ we set other is a points $y > c$ in $[a,b]$ such that the interval $[c,y]$ can be covered by atmost two elements of $A$.

Applying step ii, we have finitely many elements of $A$. &3

$c \in \mathcal{C}$, then by defn. of $\mathcal{C}$ $[a,c]$ can be covered by finite number of elements of $A$.

$[a,y] = [a,c] \cup [c,y]$ is also covered by finite number of elements of $A$.

By defn. of $\mathcal{C} \Rightarrow y \in \mathcal{C}$ [by ii].

Since $c =$ lub of $\mathcal{C}$, we have $y \leq c$.

This is a $\Rightarrow \Leftarrow$ to $y > c$.

Hence the interval $[a,b]$ can be covered by a finite number of elements of $A$.

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Corollary: 6.2.

PT each closed interval in $R$ is compact.

pf:-

Let $X = R$ in the previous thm, same proof.

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Thm 8.5-

Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be the given eqn. $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each $\alpha$. Let $\prod X_\alpha$ have the product topology. then the function $f$ is continuous $\Leftrightarrow$ each function is continuous.

pf:-

Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be continuous.

claim, $f_\alpha : A \to X_\alpha$ be continuous.

Consider the projection map.

$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ defined as $\pi_\beta(x_\alpha) = x_\beta$.

First claim, $\pi_\beta$ is continuous.

Let $U_\beta$ be an open set in $X_\beta$.

Then $\pi_\beta^{-1}(U_\beta)$ is a subbasis elements for the product topology in $\prod_{\alpha \in J} X_\alpha$.

$\therefore \pi_\beta^{-1}(U_\beta)$ is open in $\prod_{\alpha \in J} X_\alpha$.

Hence the function $\pi_\beta$ is continuous.

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Now, given $f : A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

Also, $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous.

$\therefore (\pi_\beta \circ f) : A \to X_\beta$ is also continuous.

Also, $(\pi_\beta \circ f)(a) = \pi_\beta(f(a))$ [$\because$ composition of two continuous func. is continuous]

$= \pi_\beta(f_\alpha(a))$

$= f_\beta(a)$

$(\pi_\beta \circ f) = f_\beta$

$\therefore f_\beta$ is continuous for every $\beta$.

Thus $f$ is continuous. [$\because (\pi_\beta \circ f)$ is continuous]

$\Rightarrow f_\alpha$ is continuous for every $\alpha$.

Coversely,

Let $f_\alpha : A \to X_\alpha$ be continuous for all $\alpha$.

Claim, $f : A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

A of different subbasis elements in the product space is of the form $\pi_\beta^{-1}(U_\beta)$.

In order to prove,

$f$ is continuous.

it is enough to prove that.

$f^{-1}(\pi_\beta^{-1}(U_\beta))$ is open in $A$.

Consider, $f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta)$

$= f_\beta^{-1}(U_\beta) \to (1)$

$f_\beta : A \to X_\beta$ is continuous. $(\because (\pi_\beta \circ f) = f_\beta)$

Thus, $f_\alpha : A \to X_\alpha$ is continuous for all $\alpha$.

$\Rightarrow f : A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

Thm: 6.3

Heine Borel thm.

A subset $R^n$ is compact. $\Leftrightarrow$ it is closed and is bounded the euclidean metric $d$ (or) in the square metric $\rho$.

Pf:

Let us prove that the following result.



Step i,

WKT, $\rho(x,y) \leq d(x,y) \leq \sqrt{n}\,\rho(x,y)$.

Step 2:

claim A Set is bounded Metric $d \Leftrightarrow$ it is bounded in the square Metric.

Pf:

Let A be bounded under the Metric d. $\forall x,y \in A$.

$d(x,y) \leq K$ for some $K>0$.

By step i, $\rho(x,y) \leq d(x,y) \leq K$.

$\rho(x,y) \leq K \;\forall\; x,y \in A$.

A is bounded in the Metric $\rho$.

Conversely,

Let A be bounded in $\rho$.

Then $\rho(x,y) \leq K$, for some $K>0 \;\forall\; x,y \in A$.

Again, Step i, $d(x,y) \leq \sqrt{n}\,\rho(x,y)$

$\therefore d(x,y) \leq \sqrt{n}\,K_1 = M$ (say)

$\therefore d(x,y) \leq M$ for some $M>0 \;\forall\; x,y \in A$.

$\therefore$ A is bounded in the Metric d.

$\therefore$ A set A is bounded in $\rho \Leftrightarrow$ A is bounded Metric in the Metric d.

Also WKT the topological induces the Metrics d and $\rho$ are the same.

Step 3:

Proof of Main thm.

Let A be a Compact subset of $R^n$.

Claim, A is closed and bounded subset of $R^n$.

Since any Metric space in $T_2$- space (by thm. 5.3) and $R^n$ is closed $\rightarrow$ ①

Consider the Collection of $(B_\rho(0,M) / M \in Z_+)$ of open subset of $R^n$ whose union is all of $R^n$.

This Collection of $B_\rho$ also cover the subset A of $R^n$.

$\therefore$ A is Compact, a finite subcollection of the covers.

It follows that,

$A \subset B_\rho(0, M)$ for some $M \in Z_+$.

Let $x, y \in A$.

Then $x, y \in B_\rho(0, M) = \rho(x, y) < 2M$.

$\Rightarrow$ The set $A$ is bounded in the metric $\rho$. $\rightarrow (2)$

By (1) & (2),

A is closed and bounded.

Conversely,

Suppose $A$ is closed and bounded.

Now $A$ is bounded under $\rho$.

$\Rightarrow$ there exists a positive number $N$ such that $\rho(x, y) < N \ \forall \ x, y \in A$.

Choose a point $x_0 \in A$ and

Let $\rho(x_0, 0) = b$.

Consider $\rho(x, 0) \leq \rho(x, x_0) + \rho(x_0, 0)$

$\leq N + b \ \forall \ x \in A$.

Let $P = N + b$, then $\rho(x, 0) \leq P \ \forall \ x \in A \subset R^n$

$x \in [-P, P]^n$

$A \subset [-P, P]^n$

WKT, any closed interval is compact.

$[-P, P]$ is compact.

$\Rightarrow [-P, P]^n$ is also compact.

[$\because$ finitely many product of the compact space is compact] th 5.2

$\therefore$ is also compact.

<u>Ex</u>:

1, The unit sphere $S^{n-1}$ in $R^n$ is compact. Because it is closed is bounded.

2, The closed unit ball $B^n$ in $R^n$ is also compact.

3, The set $A = \{x \times 1/x \, / \, 0 < x < 1\}$ is closed in $R^n$ But it is not compact. It is not bounded.

4, The set $S = \{x \times \sin 1/x \, / \, 0 < x \leq 1\}$ is bounded in $R^2$. but it is not compact it is not closed.

Thm 6.4

(*) Extreme Value thm (or) Maximum and Minimum Value thm.



Let $f: X \to Y$ be continuous. Where $Y$ is an ordered set in the ordered topology. If $X$ is compact then there exists a point $c$ and $d$ of $X$ such that $f(c) \le f(x) \le f(d)$.

pf:

Given $f: X \to Y$ is ~~are~~ continuous and $X$ is compact.

$f(X) = A$ is compact. [$\because$ Continuous image of a compact space is compact]

Claim,

A has largest element $M$ and smallest element $m$ such that $m = f(c)$ and $M = f(d)$ for some point $c$ and $d$ of $X$.

Suppose A has no largest element.

Then the collection $\{(-\alpha, a) / a \in A\}$ form an open cover to A.

Since A is compact.

This collection has finite sub collection $\{(-\alpha, a_1), (-\alpha, a_2), \dots (-\alpha, a_n)\}$ to cover A.

$$A = \bigcup_{i=1}^{n} (-\alpha, a_i) \to (1)$$

Let $a_j$ be the largest element of $a_1, a_2 \dots a_n$.

Now, $a_j \notin (-\alpha, a_i)$, $i = 1$ to $n$ but $a_j \in A$.

Which is $\Rightarrow \Leftarrow$ to (1)

$\therefore$ A has a largest elements.

$||^{ly}$, we can show A has a smallest element.

Thm 7.1

Compactness implies limit point compactness but not conversely.

pf:-

Let $X$ be a compact space.

claim,

$X$ is a limit point compactness.

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ie, To prove, every infinite subset A of X has a limit point.

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We prove this thm by contrapositive

If A has no limit point.

then A must be finite.

Let us assume that, A has no limit point.

Suppose, A contains all its limit points.

$\therefore$ A is closed.

Being A is a closed subset of the compact space X. then A is also compact.

Now,

for each a in A, choose a neighbourhood $U_\alpha$ of a in A such that $U_\alpha \cap (A - \{\alpha\}) = \phi$

$\Rightarrow U_\alpha \cap A = \{a\}$.

[$\because$ suppose not, then $U_\alpha$ intersects A in some other point other than a $\Rightarrow$ a is a limit point of A].

Which is a $\Rightarrow \Leftarrow$ A has no limit point.

$\therefore U_\alpha \cap A = \{a\}$.

Now, $\{U_\alpha\}$ will be an open covering to A.

$\therefore$ A is compact and $\cup U_\alpha$ covers A.

$\therefore$ We have, A has a finite subcover.

Let it be $\{U_{a_1}, U_{a_2} \dots U_{a_n}\}$.

Each $U$ contains only one element A and $A \subseteq U_{a_1}, U_{a_2} \dots U_{a_n}$

$A \subseteq \bigcup\limits_{i=1} a_i$

$\Rightarrow$ The set A contains finite number of elements namely, n-elements.

$\Rightarrow$ A is finite.

Thus, Compactness $\Rightarrow$ Limit point Compactness.

But the covers in not true.

i.e, Limit point compactness need not imply compactness.

Ex:

Consider the minimal uncountable well ordered set $S_\Omega$ in the ordered topology.

Now, $\bar{S}_\Omega$ is compact. but $S_\Omega$ is not compact. Because $S_\Omega$ is not a closed subset of the compactness space $\bar{S}_\Omega$.

However,

$S_\Omega$ is a limit point is compact.

Let A be an infinite subset of $S_\Omega$

Choose a subset B of A which is countably infinite

Being countable the set B has an upper bound say b in $S_\Omega$.

$\therefore B \subset [a_0, b]$ of $S_\Omega$ where $a_0$ is the smallest element of $S_\Omega$.

$S_\Omega$ has lub property $[a_0, b]$ is compact.

By thm 7.1,

compactness $\Rightarrow$ limit point compactness.

$[a_0, b]$ is a limit point. Compact $\Rightarrow$ B has a limit point.

Let it be x the point and is also limit point of A.

Thus every infinite subset of $S_\Omega$ has limit ~~point~~ A.

$\therefore S_\Omega$ is a limit point compactness.

Thm 7.2.

Lebesgue Number Lemma:

Let $\mathcal{A}$ be an open covering of the metric space $(X, d)$. If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than $\delta$, there exists an element of $\mathcal{A}$ containing it.

The number $\delta$ is called a Lebesgue number for the covering $\mathcal{A}$.

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Pf:

Case i,

Let $A$ be an open covering of $X$.

If $x$ is itself is an element of $A$.

Then, any positive number is a lebesgue for $A$.

$\therefore$ The proof is trivial.

Case ii,

Assume that $X$ is not an element of $A$.

$\because X$ is compact, there exist a finite subcollection $\{A_1, A_2 \dots A_n\}$ of $A$ cover $X$.

ie, $X = \bigcup_{i=1}^{n} A_i \longrightarrow (1)$

For each $i$, set $C_i = X - A_i$

Define, $f: X \to R$.

Let $f(x)$ be the avarage of the number $d(x, C_i)$

ie, $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) \longrightarrow (2)$

To prove that,

$f(x) > 0 \;\forall\; x \in X$.

Given $x \in X$

$\therefore x \in A_i$ for some $i$ [by (1)]

Define, choose $\varepsilon_i$, so the $\varepsilon_i$ neighbourhood of $x$ lies in $A_i$.

[This is possible because $A_i$ is open].

Then, $d(x, C_i) \geq \varepsilon_i$ $[B_d(x, \varepsilon) \subset A_i]$

$\Rightarrow d(x, A_i) < \varepsilon_i$

$\therefore f(x) \geq \frac{\varepsilon_i}{n}$ [by (2)]

Since $f$ is continuous it has a minimum value $\delta$, where $\delta = \min\{\frac{\varepsilon_1}{n}, \frac{\varepsilon_2}{n} \dots \frac{\varepsilon_n}{n}\}$

To show, $[\because \delta \leq \frac{\varepsilon_i}{n}]$

$\delta$ is our required Lebesgue number for the covering $A$.

Let $B$ be a subset of $X$ having diameter less than $\delta$. ie, $B \subset A$.

Let $x_0 \in B$.

Then, there exists a $\delta$ neighbourhood $x_0$ which contains $B$.

ie, $B \subset B_d(x_0, \delta)$ $[\because dia(B) < \delta]$

Let $d(x_0, C_m)$ = larger of $\{d(x, C_i)\}$

Define by defn. of $\delta$.

$\delta \leq \frac{\varepsilon_i}{n} \leq f(x_0) \leq d(x_0, \varepsilon_m)$

$\therefore d(x_0, C_m) \geq \delta$.

$\therefore d(x_0, y) \geq \delta. \ \forall y \in C_m$.

$y \notin B_d(x_0, \delta)$

$\Rightarrow B_d(x_0, \delta) \cap C_m = \phi$

$\Rightarrow B_d(x_0, \delta) \subset X - C_m = A_m$.

$\Rightarrow x_0 \in A_m$

$\Rightarrow B \subset A_m$.

$\therefore \delta$ is a Lebesgue number for the covering A.

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Thm 7.3

Uniform Continuity theorem.

Let $f: X \to Y$ be a continuous map of the compact metric space $(X, d_X)$ to the metric space $(Y, d_Y)$. Then $f$ is uniformly continuous.

pf:

Claim, f is uniformly continuous.

ie, claim, for given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x_1, x_2) < \delta$.

$\Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon \ \forall x_1, x_2 \in X$.

Let $\varepsilon > 0$, take the open covering of Y by balls $B(y, \varepsilon/2)$.

ie, Let $\{B(y, \varepsilon/2) / y \in Y\}$ is an open covering to Y.

$\therefore f$ is continuous $\Rightarrow A = \{f^{-1} B(y, \varepsilon/2) / y \in Y\}$ will be an open covering to X.

Given X is compact. this cover A has a Lebesgue number say $\delta > 0$.

Let $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$.

Define the $\{x_1, x_2\}$ has a diameter $< \delta$.

By lebesgue number thm,

there exist an element of A containing this set $\{x_1, x_2\}$.

$f^{-1}(B(y, \varepsilon/2))$ is the set in $A$ containing the set $\{x_1, x_2\}$ for some $y \in Y$.

i.e, $\{x_1, x_2\} \subset f^{-1}(B(y, \varepsilon/2))$

$x_1, x_2 \in f^{-1}(B(y, \varepsilon/2))$

$\Rightarrow (f(x_1), f(x_2)) \in B_d(y, \varepsilon/2)$

$\Rightarrow d_Y(f(x_1), y) < \varepsilon/2$ and $d(y, f(x_2)) < \varepsilon/2$.

$\therefore d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), y) + d_Y(y, f(x_2))$

$< \varepsilon/2 + \varepsilon/2$

$< \varepsilon$.

$d_Y(f(x_1), f(x_2)) < \varepsilon$.

Then $d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$. $\forall x_1, x_2 \in X$.

$\therefore f$ is Uniformly continuous.

Thm 7.4.

Let $X$ be a Metrizable space. then the following are equivalent.

i, $X$ is compact.

ii, $X$ is a Limit point. compact.

iii, $X$ is a sequentially compact.

Pf:-

(i) $\Rightarrow$ (ii) have been already proved (thm 7.1)

To prove, (ii) $\Rightarrow$ (iii)

claim, $X$ is point compactness.

$\Rightarrow X$ is sequentially compact.

Given a sequence $\{x_n\}$ of points of $X$.

Consider the set $A = \{x_n / n \in Z_+\}$

Case i,

Suppose $A$ is a finite set.

Then there exists a point $x$ such that $x_n = x$ of the sequence which is repeated infinitely many times.

$\therefore$ The sequence $(x_n)$ for a constant subsequence that is covergent automatically.

Case ii,

Suppose $A$ is infinite

Given $X$ is a Limit point compact.

83 $\therefore$ Every infinite subset has a limit point
Hence $A$ has a limit point.
Let it be $x_1$. 33
Since $x$ is the limit point of $A$.
We have every neighbourhood of $x$ intersects $A$ at infinitely many points.
Now,
Choose a positive number $n_1$ such that $x_{n_1} \in B_d(x,1) \cap A$
$\|^{ly}$, consider the ball $B_d(x, 1/2) \cap A$ and
choose $n_2 \in Z_+$ such that $n_2 > n_1$ and $x_{n_2} \in B_d(x, 1/2) \cap A$.
Continuing this process, we obtain an increasing sequence $n_1 < n_2 < \dots < n_i$ such that $x_{n_i} \in B_d(x, 1/i) \cap A$.
Claim,
This subsequence $\{x_{n_i}\}$ converges to $x$.
Let $\varepsilon > 0$, choose $i$ such that $1/i < \varepsilon$.
Let $n_k \geq n_i$ then $k \geq i$
Choose $x_{n_k} \in B_d(x, 1/k) \Rightarrow d(x_{n_k}, x) < 1/k < \varepsilon$.
$\therefore$ $X$ is sequently compact.
Next to prove, iii) $\Rightarrow$ i)
Assume that $X$ sequentially compact.
To prove,
$X$ is compact.
Step 1,
First we prove that, for every $\varepsilon > 0$.
there exist a finite covering of $X$.
ie, To prove that, $X$ can be covered by finite $\varepsilon$-open balls.
Now, let us construct a sequence of points $x_n$ in $X$ as follows.
Assume $X$ is cannot be covered by finite many $\varepsilon$-ball.
Let $x_1$ be any points of $x$.
[$\because$ $X$ is not covered by finitely many $\varepsilon$-ball].

$x$ is not covered by a single ball $B(x_1, \varepsilon)$.

$\therefore$ There exists $x_2 \in X$ such that $x_2 \notin B(x_1, \varepsilon)$. 

Now, consider $B(x_2, \varepsilon)$

Again $x$ is not covered by $B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$

$\therefore$ there exists $x_3 \in X$ such that $x_3 \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$

Choose, $x_{n+1} \in X$ and $x_{n+1} \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \cdots B(x_n, \varepsilon)$.

$\therefore d(x_{n+1}, x_i) \geq \varepsilon \quad \forall \; i = 1$ to $n$.

Hence Clearly the sequence $\{x_n\}$ Cannot have a Convergent subsequence.

Which is a $\Rightarrow \Leftarrow$ to $X$ is sequentially compact.

$\therefore$ for every $\varepsilon > 0$, $X$ can be Covered by finitely many $\varepsilon$-ball.

Step ii,

Let $A$ be any open covering of $X$.

To prove that,

$X$ is compact.

ie, To prove, there exists a finite sub Collection in $A$ which also Cover $X$.

$\therefore A$ is open covering of $X$ and $X$ is Sequentially compact.

By lebesgue number lemma,

$A$ has a Lebesgue number $\delta$, take $\varepsilon = \delta/3$ $\frac{\delta}{3}$

By step i, there exists a finite $\varepsilon$-ball covering of $X$.

Let us denote the balls by $B_1, B_2 \ldots B_K$

By lebesgue number lemma,

This diameter of each of these balls less than $2\delta/3 < \delta$.

Each of these balls is contained in some number of $A$.

ie, $B_i \subseteq A_i$ ; $A_i \in A \quad \forall \; i = 1$ to $K$.

$X \subset \cup B_i \subset \cup A_i$, $i = 1$ to $K$.

$\{A_i\}$, $i = 1$ to $K$ be finite subfamily of $A$ which covers $X$.

$\therefore X$ is Compact.

# TOPOLOGY

# UNIT 5

Unit-V

The countability axioms - the separation axioms - the Urysohn's lemma - the Urysohn's metrization theorem.

## Unit-5

Defn:-

A topological space $x$ is said to satisfy the second countability axiom if $x$ has a countable basis for its topology.

Thm 12:-

A subspace of a first countable space is first countable and a countable product of first countable spaces is first-countable. A subspace of second countable space is second-countable and a countable product of second countable spaces is second countable.

Pf:-

Consider the second countability axiom
If $\mathcal{B}$ is a countable basis for $x$.

Then $\{B \cap A / B \in \mathcal{B}\}$ is a countable basis for the subspace $A$ of $x$.

If $\mathcal{B}_i$ is a countable basis for the space $x_i$.
then the collection of all products $\pi U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values of $i$ and
$U_i = x_i$ $\forall$ other value of $i$ is a countable basis for $\pi x_i$

The proof for the first countability axiom is similar.

Thm : 13

Suppose that $x$ has a countable basis. Then,
a, Every open covering of $x$ contains a countable subcollection covering $x$.
b, There exists a countable subset of $x$ that is dense in $x$.

Pf:-

Let $\{B_n\}$ be a countable basis for $x$. Let $\mathcal{A}$ be an open covering of $x$.
For each positive integer $n$ for which it is possible,

a

Choose an element $A_n$ of $\mathcal{A}$ containing the basis element $B_n$. 2

The collection $\mathcal{A}'$ of the sets $A_n$ is countable.

Since it is indexed with a subset $J$ of the positive integer.

Furthermore,

It covers $X$.

Given a point $x \in X$, we can choose an element $A$ of $\mathcal{A}$ containing $x$.

Since $A$ is open.

there is a basis element $B_n$ such that $x \in B_n \subset A$.

Because $B_n$ lies in an element of $\mathcal{A}$.

The index $n$ belongs to the set $J$, so $A_n$ is defined;

Since $A_n$ contains $B_n$, it contains $x$.

Thus $\mathcal{A}'$ is a countable subcollection of $\mathcal{A}$ that covers $X$.

b)

From each nonempty basis element $B_n$, choose a point $x_n$.

Let $D$ be the set consisting of the points $x_n$. $D = \{x_n / n \in Z_+\}$

Then $D$ is dense in $X$.

Given any point $x$ of $X$

every basis element containing $x$ intersects $D$.

So $x$ belongs to $\bar{D}$.

<u>Defn:</u>

A subset $A$ of a space $X$ is called dense in $X$ if $\bar{A} = X$.

Lemma 2.1

(i) Let $X$ be a topological space. Let one point sets in $X$ be closed.

a, $X$ is regular. iff given a point $x$ of $X$ and a neighbourhood $U$ of $x$ there is a neighbourhood $V$ of $x$ such that $\bar{V} \subset U$.

b, $X$ is normal iff given a closed set $A$ and an open set $U$ containing $A$, there is an open set $V$ containing $A$ such that $\bar{V} \subset U$.

pf:

suppose that $X$ is regular.

Suppose that, the point $x$ and the neighbourhood $U$ of $x$ are given.

Set $B = X - U$, then $B$ is closed set.

By hypothesis,

there exist disjoint open sets $V$ and $W$ containing $x$ and $B$ respectively.

The set $\bar{V}$ is disjoint from $B$.

Since if $y \in B$.

The set $W$ is a neighbourhood of $y$ disjoint from $V$.

$\therefore \bar{V} \subset U$, as desired.

To prove,

the converse.

Suppose the point $x$ and the closed set is not containing $x$ are given.

Let $U = X - B$.

By hypothesis,

there is a neighbourhood of $x$ such that $\bar{V} \subset U$.

The open sets $V$ and $X - \bar{V}$ are disjoint open sets containing $x$ and $B$ respectively.

Thus $X$ is regular.

b) This proof used exactly the same argument.

One just replaces the point $x$ by the set $A$ throughout.

Thm 22

a, A subspace of a Hausdorff space is Hausdorff. a product of Hausdorff space is Hausdorff.

b, A subspace of a regular space is regular. a product of regular spaces is regular.

Pf:-

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a, Let $X$ be Hausdorff space.

Let $x$ and $y$ be two points of the subspace $Y$ of $X$.

If $U$ and $V$ are disjoint neighbourhood in $X$ of $x$ and $y$ respectively.

Then $U \cap Y$ and $V \cap Y$ are disjoint neighbourhood of $x$ and $y$ in $Y$.

Let $\{X_\alpha\}$ be a family of Hausdorff space.

Let $X = (x_\alpha)$ ~~be a family~~ and $Y = (y_\alpha)$ be disjoint points of the product space $\pi X_\alpha$.

Since $X \neq Y$.

there is some index $\beta$ such that $X_\beta \neq Y_\beta$.

Choose disjoint open sets $U$ and $V$ in $X_\beta$ containing $x_\beta$ and $y_\beta$ respectively.

Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\pi X_\alpha$ containing $\overset{X}{x}$. and $\overset{Y}{y}$ respectively.

b, Let $Y$ be a subspace of the regular space. Since,

$Y$ is Hausdorff, one point sets are closed in $Y$.

Let $x$ be a point of $Y$ and let $B$ be a closed subset of $Y$ disjoint from $x$.

Now,

$\bar{B} \cap Y = B$, Where $\bar{B}$ denotes the closure of $B$ in $X$

$\therefore x \notin \bar{B}$

So, using regularity of $X$, We can choose disjoint open sets $U$ and $V$ of $X$ containing $x$ and $B$ respectively.

Let $\{X_\alpha\}$ be a family of regular spaces.

Let $X = \pi X_\alpha$, By (a), $X$ is Hausdorff.

So that, one point sets are closed in $X$.

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We use the preceeding lemma,

To prove regularity of $X$,

Let $x = (x_\alpha)$ be a point of $X$ and let $U$ be a neighbourhood of $x$ in $X$.

Choose a basis element $\pi U_\alpha$ about $x$ contained in $U$.

Choose for each $\alpha$,

a neighbourhood $V_\alpha$ of $x_\alpha$ in $X_\alpha$ such that $\bar{V_\alpha} \subset U_\alpha$.

If it happens that $U_\alpha = X_\alpha$.

Choose $V_\alpha = X_\alpha$ then $V = \pi V_\alpha$ is a neighbourhood of $x$ in $X$.

We assert that,

$\bar{V_\alpha} = \pi \bar{V_\alpha}$. It follows that $\bar{V} \subset \pi U_\alpha \subset U$.

So that, $X$ is regular.

To prove,

The assertion, we show that, if $A_\alpha \subset X_\alpha$ for each $\alpha$ and if $A = \pi A_\alpha$. then $\bar{A} = \pi \bar{A_\alpha}$.

Suppose that,

$y = (y_\alpha)$ is in $\pi \bar{A_\alpha}$.

Let $U = \pi U_\alpha$ be a basis element containing $y$.

Since $y_\alpha \in \bar{A_\alpha}$, the open set $U_\alpha$ must intersects $A_\alpha$.

So, we can choose a point $z_0 \in U_\alpha \cap A_\alpha$ for each $\alpha$.

Then, $U$ intersects $A$ in the point $z = (z_\alpha)$.

Thus, $y$ is in $\bar{A}$.

Conversely,

Suppose that, $y$ is in $\bar{A}$.

We show that,

for any given index $\beta$.

We have $y_\beta \in \bar{A_\beta}$

Let $U_\beta$ be a neighbourhood of $y_\beta$.

then $\pi_\beta^{-1}(U_\beta)$ is a neighbourhood of $y$.

So that,

it intersects $A$ in some point $z$.

then, $U_\beta$ intersects $\pi_\beta(A) = A_\beta$, the point $\pi_\beta(z)$.

Thus, $y_\beta$ is in $\bar{A_\beta}$.

c, Finding an example of a subspace of a normal

# The End